APPLICATION OF OPTIMIZATION TECHNIQUES TO MILITARY PETROLEUM PROBLEMS

DEAN L. KELLOGG

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by

Dean L. Kellogg

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TABLE OF CONTENT

TABLE OF CONTENT (continued)

LIST OF FIGURES

LIST OF TABLES

INTRODUCTION

One of the significant developments of modern times has been the rapid growth in the size and complexity of public and private organizations. Decisions made by military and industrial leaders affect large amounts of capital and numerous people. Errors can be extremely expensive and an incorrect decision may require years to rectify. Further, decisions must be made quickly to match the increasing speed and change of modern human endeavors. Failure to act rapidly may provide a competitor with a marked advantage.

Since World War II, men have turned increasing effort toward improving the speed and accuracy of decision making by application of analytical and objective methods. The techniques for optimizing decisions presented in this thesis, represent an important part of those methods

The optimization techniques explored here have potentially a wide application in the field of petroleum management. These are: linear programming, game theory, and inventory theory. Chapter III includes a brief discussion of queueing theory and Chapter IV concerns the optimization of purchasing policies.

The treatment of each field endeavors to include a survey of the extensive literature covering the technique and provides an evaluation of appropriate source material. The techniques are presented in some detail with specific references suggested where it is considered that the reader may

desire exhaustive treatment of a particular facet. Proofs are not offered but are generally available in the references. The application of each technique to various military and industrial petroleum problems is handled in the form of assumed problems or as a discussion of applications believed appropriate to petroleum problems facing the military establishment.

Before proceeding, it is considered appropriate to describe the general characteristics which are common to each of the optimization techniques covered in this thesis. The methods discussed are those which seek to optimize the effectiveness of operating systems.

Measure of Effectiveness

In order to compare the effectiveness of various alternative courses of action, a scale of some type, usually numerical, must be selected as a measure of effectiveness. This may be in units of dollars, hours, successes per hundred attempts, barrels per day, or it may be dimensionless, such as the probability of destruction of a target. The effectiveness may depend on a number of variables, some of which are beyond the control of the decision maker. All factors must be accounted for in a model which serves to relate the various alternatives to the effectiveness of the system.

Model Formulation

Basic to the solution of a problem involving an operating system, is the development of a model. Models can be

highly mathematical - and either deterministic or probabalistic - or they can be qualitative and non-mathematical. The primary function of a model is to provide a means of bringing together the facts and data bearing on the problem. As a substitute for the real system, the model can be manipulated to test various alternatives in a way which an actual system can rarely be handled. The degree of correlation between the actual system and its model is a partial measure of the limitations which must be recognized in the solution.

Sub-Optimization

It is possible to optimize the cost of operation of a catalytic cracking unit but to do so at the expense of the overall long term profits of the refinery. The selection of the proper level at which to optimize operations, is an important decision. Optimization at too low a level leads to sub-optimization. At the same time, the problem at a higher level may become so complex that an adequate solution becomes extremely difficult if not impossible.

The optimization techniques presented in this thesis include an important segment of the analytical and objective methods which are being increasingly applied by government and industry to achieve better decisions.

CHAPTER I

LINEAR PROGRAMMING

BACKGROUND

The techniques of linear programming have been utilized extensively and with considerable success in many instances. It should be noted that such techniques are applicable to some, but not all of the problems encountered in programming. Among the numerous operating systems, there are many in which the fundamental relationships of the variables are linear in character. The scientific study of such systems, with the aim of optimizing selected factors, has led to the development of the theory of linear programming.

A number of authorities have indicated that linear programming is related to and developed from economic theory. Historically, VonNeumann, Neisser and others recognized during the 1930's that the simple version of the Walrasian general equilibrium could not be adequately treated in terms of the number of equations and unknowns. More recent interpretations of the early work of Frederick Taylor, Smith and Walras by economists such as Samuelson, gave impetus to the development of linear programming.

In 1945, George Stigler developed a solution to the diet problem which involved determination of adequate diet at minimum cost for 77 foods and 9 nutrients. He determined that optimum diet in 1939 consisted of wheat flour, cabbage and dried Navy beans and cost $39.93 for the year[1].

Organized research in the area of linear programming was conducted in the U.S. Air Force Project SCOOP (Scientific Computation of Optimum Programs). The outstanding contribution of this project was made by George B. Dantzig in 1947 when he set forth the mathematical statement of the general linear programming problem and developed a means of solution known as the simplex method. Prior to this time, many problems had been unsolvable even though recognized as linear programming problems[2].

Other contributions were made by Hitchcock (1941) and Koopmans (1947) who independently developed the transportation problem and by Charnes who extended the application of the simplex method.

The difficulty of solution of long linear programming problems by manual calculation has led to the use of electronic computers which add nothing to the theory but provide tremendous calculating speed. The first successful use of a computer in the solution of a linear programming problem was on the Bureau of Standards SEAC in 1952[3]. Most electronic computers have now been programed to solve the simplex algorithm and linear programming has become a powerful tool in the optimization of operating systems.

EVALUATION OF BIBLIOGRAPHY

A very real problem exists in selecting sources of information for preparation of a paper on linear programming methods and applications. While the material available on

the subject has sprung forth only in the years since World War II, the volume of articles, books and studies in this area has become a veritable flood. It has not been possible to review more than a portion of the books and articles involving linear programming, but a conscientious effort has been made to select those which appear to be applicable in the light of my subject. The 865 page annotated bibliography, Operations Research by Batchelor[4], has been of considerable assistance and is recommended to those seeking sources on methods and applications.

The basic work in linear programming done by Dantzig, Charnes, Koopmans, and Kuhn has led the author to utilize works by these men in the theoretical areas where possible. However, the developments of these men are effectively presented in more readable form in the basic text covering linear programming by Gass[5]. For those interested essentially in learning and applying the techniques of linear programming, the book by Gass is recommended as well as the sections on linear programming by Saaty[6] and Churchman, Ackoff and Arnoff[7].

For applications to military and industrial petroleum problems, the work done by Alan S. Manne[8,9], and Gifford H. Symonds[10] is excellent in the areas of refinery operation. For applications involving tanker scheduling and other aspects of the transportation problem, the work done by Dantzig and Fulkerson[11] as well as Flood[12], provides an excellent basis for further study.

For the reader interested in additional research in the area of linear programming, it is suggested that a review be made of the bibliography on linear programming by Vera Riley and Saul Gass[13].

Mathematical Methods of Linear Programming

Programming is the planning of an operation for the purpose of optimizing the result. The objective may be to minimize costs or to maximize returns. It may also be to allocate efficiently limited or constrained resources to meet specific objectives. The number of solutions may be infinite, but the best or optimum solution must satisfy both the conditions or constraints of the problem as well as the objective.

When linear (straight line) constraints are assumed together with a linear objective, optimization requires the solution of a linear programming problem. Assumptions of linearity are frequently appropriate, however, it may be found that the objective function is non-linear and that use of such a function improves the result. It is considered that non-linear programming is outside the scope of this thesis and no treatment of the subject is included. Extensive information on this subject may be found in 14 and 15.

The General Linear Programming Problem

The mathematical model of the general linear programming problem consists of a set of simultaneous linear inequations which represent the conditions or constraints of the problem

and a linear function which describes the objective of the problem. Mathematically a linear programming problem may be formulated by letting a_{ij}, b_i, and c_j be sets of known coefficients with $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ and also letting $x_j (j = 1, \ldots, n)$ be a set of unknown variables.

In the general linear programming problem, we seek solution sets which may be regarded as vectors, $X = (x_1, x_2, \ldots, x_n)$, which satisfy the linear inequalities previously referred to as constraints, and at the same time maximize the linear objective function.

The inequalities in the general case are of the form:

$$\sum_{j=1}^{n} a_{ij}\, x_j \leqq b_i \qquad (i = 1, \ldots, m) \qquad (1\text{-}1)$$

The foregoing is a condensed notational form which can be more easily understood in expanded form as shown here:

$$a_{11}\, x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leqq b_1$$

$$a_{21}\, x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leqq b_2$$

$$\vdots$$

$$a_{m,1}s_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leqq b_m$$

It is also established that the values of the variables are positive and the solution set is thus subject to the constraint:

$$x_j \geqq 0 \qquad (j = 1, \ldots., n) \qquad (1\text{-}2)$$

The linear objective function which is to be maximized is of the general form:

$$Z = \sum_{j=1}^{n} C_j x_j \qquad (1\text{-}3)$$

The objective function can be expanded in the more easily understood form:

$$Z = C_1x_1 + C_2x_2 + \ldots + C_jx_j$$

Before attempting to illustrate a solution, let us examine the problem structure briefly. Equation (1-3) defines the objective function, Z, which is linear in each value of x_j. It is a function of the vector $X = (x_1, x_2, \ldots, x_n)$ and may be expressed as:

$$Z = f(x) \qquad (1\text{-}4)$$

This function is defined for all values of x with finite components, but must also satisfy the restrictions of equations (1-1) and (1-2).

Equation (1-2) is usually called the non-negativity restriction and requires that only positive values of x be considered. Equation (1-1) requires that feasible solutions satisfy m linear inequalities generally termed "constraints".

The linear programming problem may be stated in an equivalent form where a linear function is to be minimized and the constraints are $\geqq$ instead of $\leqq$. Since this problem is obtained from the general problem described by multiplying the inequalities by -1 and maximizing -Z, this case can be covered by limiting this discussion to the maximization problem.

Methods of Solution

The primary methods of solution of the general linear programming problem are the geometric method, the relaxation method, the method of double description and the simplex method. Consideration of the geometric method will serve to illustrate and clarify the problem and is therefore included at this point. A discussion of the simplex method, which is the most frequently used technique, will also be given.

Geometric Method

Geometrically, a solution to the linear programming problem is a point of the convex set defined by the constraints which also maximizes (minimizes) the objective function.

To illustrate, let us consider a simple production problem of a manufacturer of two types of lubricating oil. Both items require the use of common equipment and labor. The objective of the company is to maximize profits on the two products. It is assumed that a fixed market price is established and that there are six fixed resources to be allocated to the production of the two items. These are the hours of skilled and unskilled labor available, the number of machine hours available on two blending machines through which both products must pass and two items of blending stock required as raw materials.

Let x_1 and x_2 be the respective quantities of the two lubricants to be produced. Assuming 5 and 8 to be the

respective unit profits, our goal is to maximize the objective function

$$Z = 5x_1 + 8x_2$$

The constraints are expressed as:

(1)	$x_2 \leqq 3$	Skilled labor
(2)	$x_1 + 2x_2 \leqq 7$	Unskilled labor
(3)	$2x_1 + x_2 \leqq 8$	Machine 1
(4)	$3x_1 + x_2 \leqq 13$	Machine 2
(5)	$4x_1 + 3x_2 \leqq 40$	Blending Stock A
(6)	$3x_1 + 2x_2 \leqq 20$	Blending Stock B

and non-negativity restrictions are

$$x_1 \geqq 0$$

$$x_2 \geqq 0$$

Because of the non-negativity restrictions we are concerned only with the first quadrant and the planes of the constraints appear as lines in two dimensions. See Figure 1. The solution set which satisfies the conditions of the constraints is thus a polygon as shown and is both bounded and convex. We are thus assured of at least one optimal solution and there are alternate optimal programs only if the objective function is parallel to one of the sides of the polygon. It will be noted that the only constraints which affect the solution set are those of skilled and unskilled labor and machine hours on the first machine only. All other constraints are redundant and of no effect.

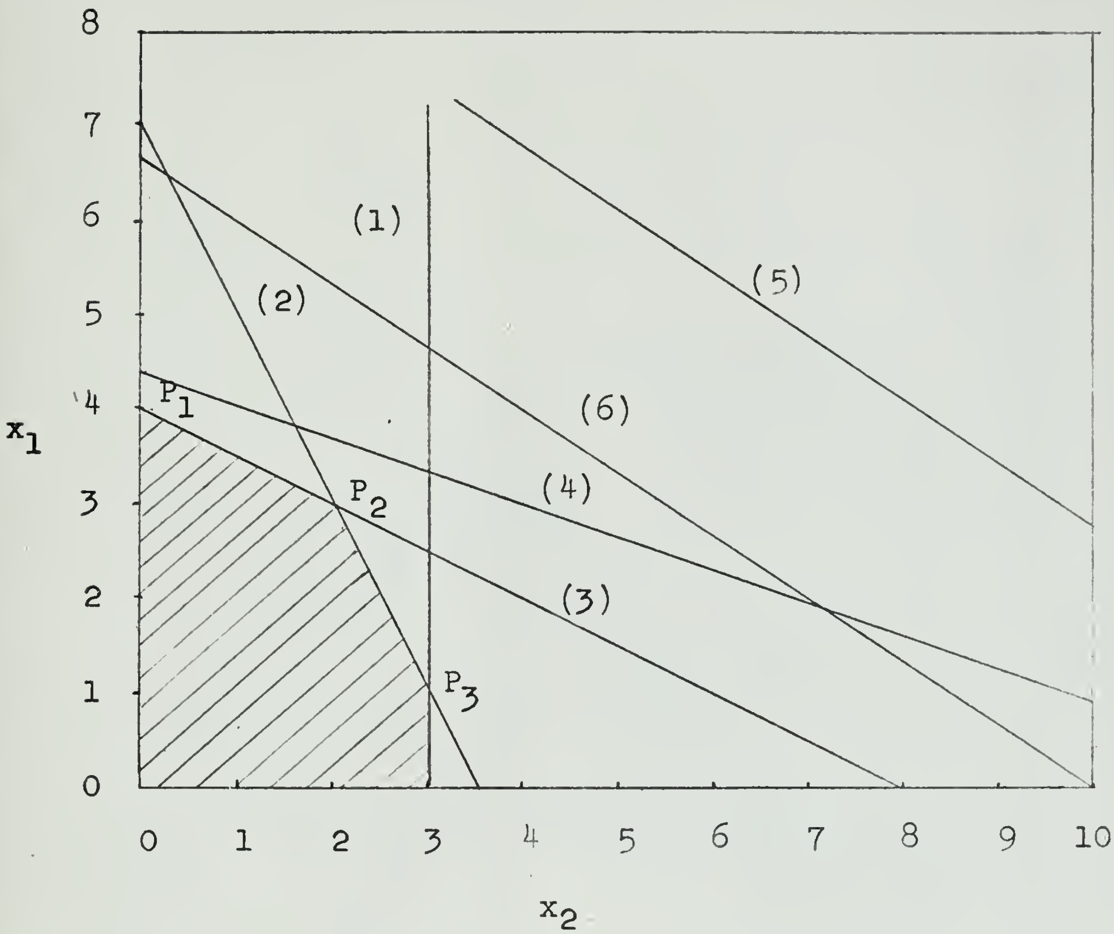

Figure 1

Set of Solutions

Figure 2 illustrates the geometric method by considering a family of planes for different values of the objective function. It will be noted that the maximum Z, being the greatest distance from the origin, is a line through P_2 with $Z = 31$ and a slope of $- 8/5$. The optimum solution thus calls for production of 2 units of x_2 and 3 units of x_1. An analysis of the solution shows that even the hours of skilled

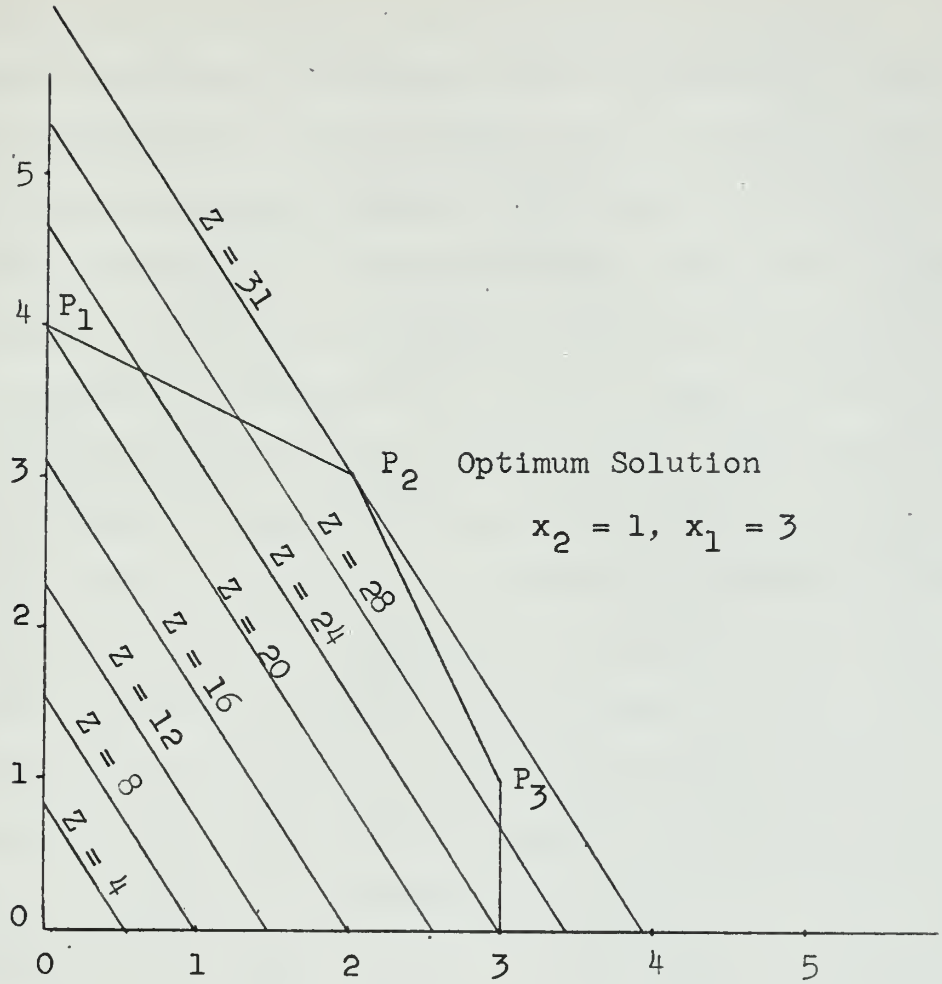

Figure 2

The Optimum Solution

labor available is not limiting, and further savings could be made by adjusting inventories and resources to the end that all constraints will pass through the point P_2, the optimum solution.

Simplex Method

The simplex method as developed by Dantzig provided the first practicable and simple system for solving linear

programming problems with large numbers of variables. As originally proposed the method did not adequately handle certain degenerate cases; however, work by Charnes resolved these problems and the simplex method was made workable without the restriction of non-degeneracy. The most complete coverage of the simplex method is set forth by Charnes[16]. This treatment is exhaustive. For the reader interested in simply learning to apply the technique, the presentations given by Gass[17], Churchman[18], and Sasieni[19] are recommended.

The simplex method is an iterative procedure and each iteration consists of translating the plane of the objective function parallel to itself for successive values of Z and evaluating its distance from the origin at the vertex of the convex set. This can be more adequately explained by referring to figure 2. Although the figure is in only two dimensions, the convex set can be visualized as describing a polyhedron formed by the planes of the constraints. One seeks the point of the polyhedron (generally on the boundary and usually at a corner or vertex) through which the Z plane may pass and which plane corresponds to a maximum value of Z. In the simplex process planes of Z are passed successively through the corners or extreme points of the polyhedron, advancing away from the origin with increasing values of Z, until a maximum (optimal) solution is obtained.

The simplex process also utilizes a criterion which eliminates some of the points as possible solutions. The number of iterations required varies but is frequently about twice the number of inequalities.

The first step in the simplex method is to convert the system of m linear inequalities to a system of m linear equations. This is accomplished by the insertion of "slack variables", x_{n+1}, x_{n+2}, ..., x_{n+m}, one for each of the inequalities which then becomes:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} && = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + \quad + x_{n+2} && = b_2 \\
&\cdot \\
&\cdot && \qquad (1\text{-}5) \\
&\cdot \\
&a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \qquad + x_{n+m} && = b_m
\end{aligned}$$

The use of the slack variables above is simply a device used in arriving at an optimal solution. The use of these slack or idle factors will be more readily understood in the numerical example to be given later in this chapter.

Having converted the inequalities to equalities (1-5), the problem can be restated as one of finding among all sets of points (x_1, x_2, ..., x_{n+m}), that point which satisfies the constraints (1-5) with non-negative values of x_j and also maximizes the objective function:

$$Z = C_1x_1 + C_2x_2 + \dots + C_nx_n \qquad (1\text{-}6)$$

The basic theorem behind the simplex technique states, in effect, that a set (x_1, x_2, ..., x_{n+m}) which maximizes the objective function and satisfies the constraints, must necessarily have n elements which are zero. If the coordinates were known, we could substitute their values in the

constraints and would then have m equations and m unknowns which could be solved algebraically for the optimal solution. Without this knowledge, it is necessary to proceed step by step to a solution.

The procedure may be explained as follows: Choose n of the n + m variables and give them a value of zero. Utilizing these values, the constraints may then be solved for the remaining m variables. Solutions with negative values for some variables are discarded as violating the non-negativity restrictions and another set of n variables is selected for trial as zero variables. When the values of all coordinates or variables of the solution set are non-negative, we have a feasible solution. This is the starting point of the iterative procedure.

The simplex method also permits a test of the feasible solution for optimality, and, if not optimal, the method allows one to proceed directly to a new feasible solution with an improved Z. Ultimately the procedure leads to an optimum solution.

Application of the simplex technique involves extensive use of matrix representation and manipulation of the rows in a series of steps until a solution can be read from it. To illustrate the manner in which this is done, a numerical example will be presented here.

Numerical Example of Simplex Method

In this example, the goal is to maximize the objective function:

$$Z = 3x_1 + 5x_2 + 4x_3 \qquad (1\text{-}7)$$

subject to the linear constraints:

$$\begin{aligned} 2x_1 + 3x_2 &\leqq 8 \\ 2x_2 + 5x_3 &\leqq 10 \qquad (1\text{-}8) \\ 3x_1 + 2x_2 + 4x_3 &\leqq 15 \end{aligned}$$

Step One. Use of Slack Variables

By introduction of "slack variables", which essentially are additional variables, the inequalities become equalities. The slack variables are non-negative and take the form, x_4, x_5, and x_6 in this example. The inequalities (1-8) thus become the following equations:

$$\begin{aligned} 2x_1 + 3x_2 \qquad + x_4 \qquad\qquad &= 8 \\ 2x_2 + 5x_3 \qquad + x_5 \qquad &= 10 \qquad (1\text{-}9) \\ 3x_1 + 2x_2 + 4x_3 \qquad\qquad + x_6 &= 15 \end{aligned}$$

The first goal of the simplex method is to obtain a feasible solution associated with an extreme point of the solution set. Such a feasible solution is available after the first "tableau" or matrix of coefficients has been formed. However, it will be necessary to continue to examine other extreme points in a series of tableaus until a maximum solution is obtained.

Equation (1-9), can now be written in matrix form, marking the columns of coefficients as P_1, P_2, ... P_6 and using P_0 for the column of constants. Below the matrix is row C_j where the coefficients of the objective function are entered.

First Tableau

C_i	P_i	P_1	P_2	P_3	P_4	P_5	P_6	P_0
0	P_4	2	3	0	1	0	0	8
0	P_5	0	2	5	0	1	0	10
0	P_6	3	2	4	0	0	1	15
	$C_j \rightarrow$	3	5	4	0	0	0	
	Solution	0	0	0	8	10	15	(Z = 0)
	Δ_j	3	5	4	-	-	-	
$b_i/a_{i,2}$		-	-	-	8/3	10/2	15/2	
			↑ entering variable		↓ departing variable			

<u>Step Two. First Feasible Solution</u>

In the foregoing matrix, a feasible solution can be read off, so long as the non-negativity restriction is not violated. This solution has been entered in a row below the objective function.

This first feasible solution was obtained by setting the variables x_1, x_2, and x_3 equal to zero. The solution is read off by selecting the non-zero variables to be those whose columns contain a single "1", with the rest being zeros. Columns P_4, P_5, and P_6 are thus chosen and for convenience are shown in the P_i column in the box at the left. The coefficients of corresponding variables, x_i, in the objective function are inserted in the column marked C_i in the same box, again for convenience. With all variables at zero in

the constraining functions, except the slack variables, the feasible solution is $x_1 = x_2 = x_3 = 0$ and $x_4 = 8$, $x_5 = 10$, $x_6 = 15$. This is indicated in the solution row of the first tableau. The value of Z associated with this solution can be seen to be zero ($Z = 0$) since the variables x_1, x_2, and x_3 in the objective function have values of zero for this first feasible solution.

Step Three. Test for Optimality

This is accomplished by computing an evaluation, Δ_j, for each zero variable, x_j, in the solution by the formula:

$$\Delta_j = C_j - \sum_{i=1}^{i=m} a_{ij} C_i \qquad (1\text{-}10)$$

At this point in the process, where the set of non zero variables is just the set of slack variables, it can be seen that $\Delta_j = C_j$ for each j. This can be explained by pointing out that in computing Δ_j for the first variable in the solution of our example, we take the value of C_1 which is 3 and subtract from this the sum of the products of the three values in the C_i column times their corresponding values in the column designated P_1 in the matrix.

Since all values in the C_i column are zero at this stage, the products are zero. Thus the summation term is zero and:

$\Delta_1 = 3 - 0 = 3$

Similarly $\Delta_2 = 5$ and $\Delta_3 = 4$. The Δ_j values are entered in the row below the solution in the first tableau.

Interpreting the Test

If one or more of the Δ_j are positive, the solution is not optimal and we must proceed to the next step. If all of the Δ_j's are zero or negative, the solution is optimal. However, if in the latter case, some of the Δ_j's are zero while others are negative, more than one optimal solution exists having the same value for Z. If all Δ_j's are negative and none are zero, a unique optimal solution has been found.

Step Four. Computing the Entering Variable

To form the next matrix, we must find the entering variable, x_r, which is that zero variable in the feasible solution which is to be non-zero in the next iteration.

The entering variable must have the following characteristics:

1. It is zero in the present matrix
2. It has a $\Delta_j \geqq 0$
3. One of the matrix coefficients, a_{ij}, in the column x_j must be greater than zero

While several variables in a solution might be chosen, the problem solution converges quickly if the entering variable, x_r, is chosen as that one for which Δ_j is largest.

In checking the first solution it can be seen that the second variable in the solution, has the necessary prerequisites

and has the largest Δ_j. It is thus marked "entering variable" in the first tableau.

Step Five. Determining the Departing Variable

The departing variable, x_s, is that non-zero variable in the feasible solution which is to become zero in the next iteration. The departing variable is generally determined by the selection of the entering variable and by the non-negativity restrictions on the coordinates. It is first necessary to compute a quotient:

$$b_i/a_{ir} \tag{1-11}$$

for each non-zero term in the solution. The values of b_i are located in the solution while the values of a_{ir} are found in the column of the previously selected entering variable. For the first quotient we compute:

$$b_i/a_{ir} = \frac{8}{3}$$

From among the three quotients calculated, the one in column P_4, is the smaller and thus determines the departing variables. The departing variable is designated in the first tableau.

Step Six. Calculate the New Matrix

To calculate the new coefficient matrix, it is necessary to perform several row operations on the prior matrix. The need for the entering and departing variables now becomes apparent. These two variables locate a position in the matrix which is marked with a box. The new matrix is to have a 1 in this position. By dividing the elements of

the first row by 3, which is the value in the designated position, unity is obtained in that position in the new matrix shown here.

2/3	1	0	1/3	0	0	8/3
0	2	5	0	1	0	10
3	2	4	0	0	1	15

The new matrix also must contain zeros in the other positions of the designated column. This is done by subtracting appropriate multiples of the new row from the other rows of the matrix.

The second intermediate matrix then becomes:

2/3	1	0	1/3	0	0	8/3
-4/3	0	5	-2/3	1	1	14/3
5/3	0	4	-2/3	0	1	29/3

The new tableau can now be written.

Second Tableau

C_i	P_i	P_1	P_2	P_3	P_4	P_5	P_6	P_0
5	P_2	2/3	1	0	1/3	0	0	8/3
0	P_5	-4/3	0	5	-2/3	1	0	14/3
0	P_6	5/3	0	4	-2/3	0	1	29/3
	C_j	3	5	4	0	0	0	0
Solution		0	8/3	0	0	14/3	29/3	(Z=40/3)
	Δ_j	-1/3	-	4	-5/3	-	-	
b_i/a_{ir}		-	∞	- ↑	-	14/15 ↓	29/12	
				entering variable		departing variable		

Here it will be noted that the second feasible solution is again read off by selecting the non-zero variables to be those whose columns contain a single "1" with the rest zeros. This marks the beginning of another iteration. By repeating steps 1 through 6, an optimum solution will be reached. In this problem, a total of four tableaus are required in order to reach an optimum solution. The third and fourth are obtained in the same manner and are shown below.

Third Tableau

C_i	P_i	P_1	P_2	P_3	P_4	P_5	P_6	P_0
5	P_2	2/3	1	0	1/3	0	0	8/3
4	P_3	-4/15	0	1	-2/15	1/5	0	14/15
0	P_6	41/15	0	0	-2/15	-4/5	1	89/15
	C_j	3	5	4	0	0	0	
Solution		0	8/3	14/15	0	0	89/15	(Z=256/15)
	Δ_j	11/15	-	-	-17/15	-4/5		
	b_i/a_{ir}	-	4	-7/2	-	-	89/41	
		Entering variable					Departing variable	

Fourth Tableau

C_i	P_i	P_1	P_2	P_3	P_4	P_5	P_6	P_0
5	P_2	0	1	0	15/41	8/41	-10/41	50/41
4	P_3	0	0	1	-6/41	5/41	4/41	62/41
3	P_1	1	1	0	-2/41	-12/41	15/41	89/41
	C_j	3	5	4	0	0	0	
	Solution	89/41	50/41	62/41	0	0	0	(Z=765/41)
	Δ_j	-	-	-	-45/41	-24/41	-11/41	

Since in this final tableau, all Δ_j are non-positive, an optimum solution has been obtained.

Degeneracy

The problem of degeneracy exists in the simplex technique when the vectors chosen are not linearly independent. In such cases, more than n vectors pass through the same point which in turn is described by the fact that more than m of the original m + n variables are zero. Under these conditions the value of the objective function remains unchanged by each iteration of the simplex method and the procedure continues to cycle.

Degeneracy problems seldom occur in actual cases and, therefore, no attempt will be made to cover the means of solving such problems here. Gass[20] has an extensive discussion of the solution of degeneracy problems and Saaty[21] describes a technique for solution of degeneracy cases.

Duality

One other aspect of linear programming should be mentioned before proceeding to the types of problems and applications possible with linear programming. This is the presence of a dual problem which is related to the original problem both in formulation and in the existence of a solution.

The pair of linear programming problems shown here are duals of each other:

Find $X = (x_1, \ldots, x_n)$	Find $Y = (y_1, \ldots, y_m)$
such that:	such that:
$x_i \geqq 0$	$y_i \geqq 0$
$\sum_j a_{ij} x_j \leqq b_i$	$\sum_i a_{ij} y_i \geqq C_j$
$\sum_j C_j x_j = Z$ is a maximum	$\sum_i b_i y_i = W$ is a minimum

The coefficients a_{ij}, b_i and C_j are constants with equal values in both the primal and the dual. Under these conditions it is apparent that there is a close relation between the two solutions. Kuhn and Tucker[22] indicate the following properties are among those characteristic of dual programs:

a. Either both the minimization and maximization have optimal solutions or neither does.

b. A feasible vector X is optimal, only if there is a feasible vector Y such that Z = W. Similarly, a feasible vector Y is optimal if there is a feasible vector X such that Z = W for these vectors.

c. If both the primal and dual problems have optimal solutions, then max Z = min W.

These properties indicate that the maximum value of Z can be found by determining the minimum value of W, or vice versa. If W has no minimum value, it then follows that Z has no maximum. Thus in any problem in which only min W or max Z are to be determined, either the primal or the dual may be solved.

In selecting the problem to be solved, primal or dual, the one most easily solved should be used. If geometric techniques are used, the problem with the least number of variables should be chosen. The reverse is generally true for the simplex method which is well suited to the solution of the dual. Churchman[23] indicates that the number of iterations required for the simplex method is from 1 to 1 1/2 times the number of rows or constraints. Examples of conversion from primal to dual are given by Gass[24].

Special Linear Programming Problems

Of the many forms of linear programming problems, several can be grouped by type and special techniques have been developed for their solution. Among these are the assignment and transportation problems which are important enough to discuss briefly here.

The Assignment Problem

This group consists of problems where there are _n_ resources and _n_ uses to which the resources are to be applied. Each resource or origin is to be associated with one and only one use or destination and we wish to assign the associations in such a way as to maximize (or minimize) the summed effectiveness. Problems may involve supplies, personnel, or equipment to be assigned to demands or jobs. Among those who have made significant contributions in this area Hitchcock, Flood, Kuhn, and Dwyer. The techniques have been applied profitably in making optimal assignments of personnel, equipment, etc. A bibliography by Riley and Gass[25] contains references to a number of such applications. As a special case of linear programming, the assignment problem can be solved by simplified means. In an n^2 matrix, if $n = 8$, there are $8!$ or 40,320 combinations to be examined. A simplex solution could be used but a technique developed by Flood is simpler and can be utilized without use of a computer. A detailed example of this method is given in Flagle[26].

The Transportation Problem

The transportation problem is a generalization of the assignment problem. The matrix of effectiveness is no longer necessarily square; however, the problem is essentially one of selecting optimum programs for distributing a homogeneous product where s_i units are positioned at origin O_i and d_j units are required at destination D_j. The cost

of transportation of a single unit from O_i to D_j is denoted by C_{ij} and our objective is to minimize $C = \sum_{ij} x_{ij} C_{ij}$, the total cost of distribution where supply equals demand. If supply is less than demand and no preferential destinations exist, then we can assume the availability of fictitious supplies which can be distributed at no cost. This procedure makes it possible to treat the problem as if supply equaled demand. The variable x_{ij} represents the quantity of units moved from O_i to D_j and therefore $\sum_i x_{ij} = s_i$ and $\sum_i x_{ij} = d_j$. The condition of supply equals demand is denoted as $\sum_i s_i = \sum_i d_j$

Transportation problems can be solved by the simplex method and other techniques but the "transportation technique" is the one most frequently used. Several examples of the transportation problem are included in most texts on operations research. An extensive transportation problem solved by the simplex method can be found in Saaty[27].

APPLICATIONS OF LINEAR PROGRAMMING

The JP5 Jet Fuel Problem

A question which is currently a concern of the U.S. Navy is one regarding the capacity of the petroleum industry to produce JP5 jet fuel. This information is desired, not only to determine the availability of JP5 during periods of war or peace, but also to consider expanding use of this type of

jet fuel and the effect that increased demand may have on the price to be paid for JP5.

No attempt will be made to answer these questions directly, but an approach to the problem will be discussed as an application of the use of linear programming techniques to solve military petroleum problems. A survey of attempts to answer similar questions, reveals that the work done by Dr. Alan S. Manne[28] in 1956 for the Rand Corporation in developing a linear programming model of the U.S. petroleum refining industry, can well serve as a basis for solution of the questions concerning JP5. The mathematical model, constructed by Manne, was one of a series of process analysis models aimed at depicting the technological capabilities of the entire national economy. In general, the purpose of these studies was to provide quantitative answers on production capabilities and also to weigh the effects of product substitutions and alternate production processes.

The general question Manne sought to answer was: With the crude oils and refineries available in the U.S., what product-mix possibilities existed between the output of JP4 jet fuel and the output of other refinery products, and what would be the effect of a loss of a portion of refining capacity?

To answer these and other questions in regard to JP5 jet fuel requires the construction of a linear programming model covering both U.S. crude oil production and the operation of the nation's refining industry. The model must

incorporate the details of the manufacture of JP5 and all other major products: 115/145 avgas; 100/130 avgas; JP4 jet fuel; kerosene; premium mogas; regular mogas; diesel; Navy special; the several grades of fuel oil; lube oils; asphalt; coke; liquefied petroleum gases; and basic aromatic chemicals.

The JP4 jet fuel model developed by Manne, did not take into consideration the problems of geographic location or the limitations imposed by transportation, new facilities or accumulation of inventories. It is understood that further work in this area has been done by T. A. Marschak but information on this work could not be located at this writing. Manne indicates that Marschak has streamlined the technological details of the model discussed here and constructed a four region model, (East, Midwest, Gulf Coast, and West) which includes limitations on transportation. Such a model would probably give an improved representation of production, refining and transportation. Manne acknowledges that a model including provisions for location and time can be constructed but that the costs of computation were prohibitive at that time (1954).

The model constructed by Manne based calculations on a reference date, 1 January 1953, and assumed that both equipment capacities and crude oil availability were set at the maximum rate estimated for that date. He based all specifications (octane number, performance number, boiling range, aromatics content, vapor pressure and viscosity) on the

averages prevailing on that date. He further assumed that labor, power, catalysts and sulfuric acid were not limiting. The model did, however, consider the refinery inputs of isopentane, natural gasoline, tetraethyl lead, heat, and C2, C3 and C4 gases in addition to crude oil.

One of the simplifications in the model was consideration of 25 categories of crude oil only. Based upon data compiled by the U. S. Bureau of Mines, a sample of the 25 leading U. S. crude oils was selected. Because of the variations in analysis of individual crude oils, assuming a single "composite" crude would not allow for the tendency of various types of refinery equipment to utilize the most suitable crude.

It was also assumed that the various types of refining processes can be designated by broad categories but not by individual patents. No distinction was made between Fluid Catalytic Cracking of Universal Oil Products Company and the Thermofore Catalytic Cracking of the Houdry Corporation since product yields of the two processes are similar. Catalytic Cracking and Catalytic Reforming, however, are distinct and were treated separately.

The Linear Programming Model

The model developed by Manne was a conventional linear programming type with numerous constraining inequalities and an objective function to be maximized. The model allowed numerous possibilities for varying the product-mix. Choices

could be made at the three main steps of the refining sequence: atmospheric crude distillation, conversion of straight-run products into blending stocks and the blending of the end items. In general the constraints introduced in the model were in six classes: (1) equipment capacity, (2) crude oil availability, (3) refinery gases and straight-run streams, (4) converted streams, (5) gasoline and jet fuel specifications, and (6) end item requirements.

The objective function to be maximized represented a product mix which, except for JP4 in this instance, was proportional to the actual 1953 production of each item. It was the purpose of the model, then, to maximize the standard product mix, subject to producing specified amounts of jet fuel. The requirement for jet fuel can be varied in the model to coincide with the expected needs of the military in a given situation. The model, in this way, provides a substitution curve between the one group of refinery products and the jet fuel. Manne referred to these as "trade-off" curves. The matrix for the model consisted of 105 rows and 310 columns and solutions were obtained by the simplex method which was programed for computer use. No attempt will be made to reproduce the details of the model or its solution here; however, it may be informative to show the "trade-off" curves obtained and to discuss something of its interpretation.

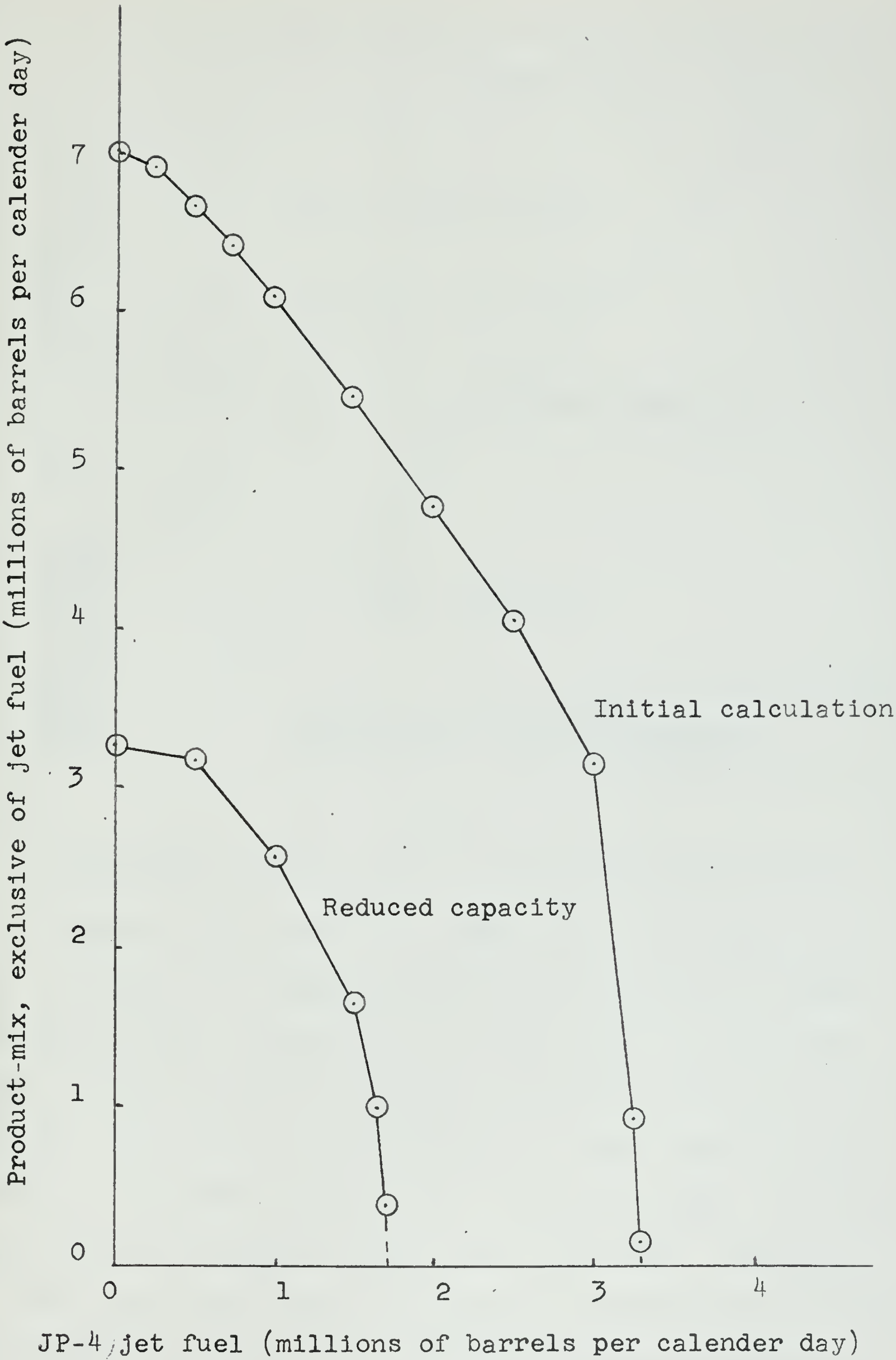

Figure 3

U. S. jet fuel trade-off curves

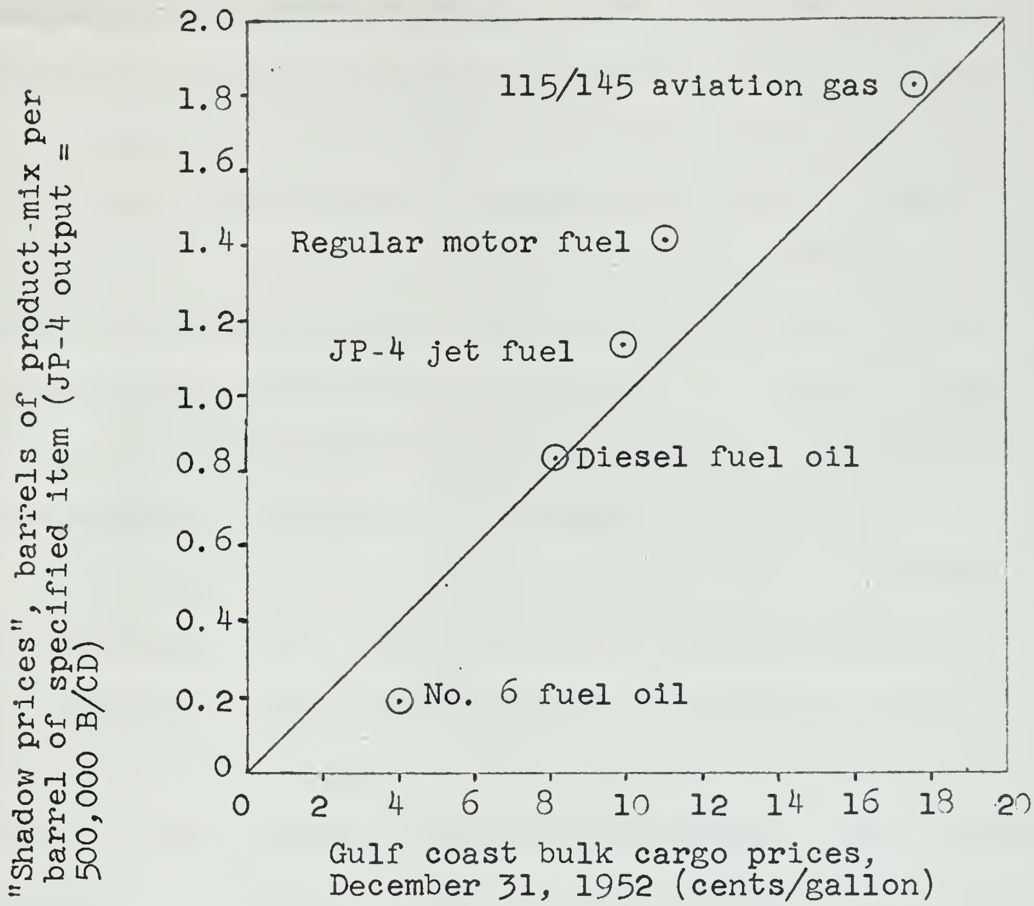

Figure 4

Comparison between market prices and shadow prices

Figure 3 represents the trade-off curve between JP-4 jet fuel and the standard-product mix - with the various assumptions and simplifications given It will be noted that, except for short segments, the relationship is not linear. The curve indicates that jet fuel cannot be substituted for other products at a constant barrel for barrel rate nor at a constant dollar for dollar rate. Examination of the curve shows that as the volume of jet fuel increases, the amount of other products which must be sacrificed becomes increasingly greater. At low volumes of jet fuel, the

trade-off curve shows that each barrel of jet fuel can be produced at a cost of about 0.6 barrels of standard product mix. However, at the extreme of about 3.29 million barrels of jet fuel, 25 barrels of product mix is sacrificed for each barrel of jet fuel. These figures should not be considered as exact amounts. Manne indicated that the curves could only be considered as estimates. In actual circumstances, industry would make adjustments and innovations not possible to include in the model.

The point shown as actual output serves to indicate that the model came fairly close to estimating actual output. However, there are some factors within the model, principally the supply of crude oil, which prevent the estimate from straying too far from the actual. Manne instituted an additional check on the model by comparing the shadow prices of end items with actual market prices at the time. A shadow price, in this model, represents the cost of an item measured in terms of the standard product-mix. Given the assumptions of the model and neglecting market factors (which certainly are present for individual products), prices should be proportional and lie on a 45° line between the two price scales As will be noted in Figure 4, while the prices were not perfectly proportional, the correlation was quite good and rough estimates of prices may be obtained so long as the prices selected for comparison are within a reasonable range of actual production volume. Such a procedure applied to a model designed to predict production of

JP5 by the refining industry, may provide a rough estimate of prices although it must be recognized that errors are inherent in such a procedure.

In a second set of calculations made by Manne, it was assumed that a reduction had been arbitrarily made in the amount of refining equipment available. The trade-off curve for the reduced equipment conditions is plotted in Figure 3. The linear programming calculations and the matrix were unchanged except for the reduced equipment conditions. Capacities in this instance were reduced to a range from 38 per cent to 51 per cent of the earlier levels. The purpose was again to maximize the level of the non-jet fuel product-mix subject to specific requirements for jet fuel. As before, the first increment of JP4 was obtained at small cost in terms of other fuels, but it increases rapidly to about 17 barrels of product-mix per barrel of jet fuel at a production rate of 1.7 million barrels of JP4. Prior to the computation, it had been expected that at reduced capacity, it would be possible to select more desirable crudes and a much larger portion of the product-mix could be converted into jet fuel. However, there was only a relatively small difference in the maximum fraction of jet fuel which could be produced from a barrel of crude. In the initial calculations this fraction reached a maximum of 45% and with reduced equipment the fraction of crude converted to jet fuel had a maximum of about 50%. A more striking difference had been expected.

It is considered that a similar model can be constructed to provide answers to questions on production of JP5. The requirements made in the model by Marschak to include the effects of location and transportation should be included as well as changes in refining processes and capacities which have occurred since 1953. A model for JP5 will be complicated by the fact that in time of war, other military fuels would be competing for a portion of the total product-mix and that industry and civilian requirements would also be altered. Nevertheless it is considered that such a linear programming model can be constructed and manipulated to provide approximate answers to some of the questions which are now unresolved. The ability of the model to adjust for reduced capacities also permits consideration of alternative actions in the event of destruction of a portion of the nations refining capacity.

On the basis of the author's review of information relating to the analysis of industry wide production and capacity problems, it is considered that linear programming offers a generally satisfactory method of solution. Other attempts to predict the capabilities of an economy have been made using gross national product which usually lead to overly optimistic results when applied to short range problems because they overstate the degree of substitutability of products. On the other hand, using a single area of production as a basis for prediction usually understates substitutability and consequently under-estimates capabilities.

The Leontief interindustry flow model was an improvement and efforts to correct the errors introduced by rigid assumptions in even this model have led to the use of linear programming as an extension of Leontief's original methods. Linear programming allows both for the production of a specific item by any of several items by a single process.

The development of an industry process analysis model is not a simple nor inexpensive task. Success requires close cooperation between the model builder, the sources of data and the numerical analyst. Perhaps the most critical area in constructing the model is in the collection of truly significant data and an understanding by the model-builder of the use of such data. While difficult, it is feasible to construct a model which will provide reasonably good results. The use and interpretation of the results then becomes the function of top management.

Additional applications of linear programming in refining operation may be found in references 29 and 30.

CHAPTER II

GAME THEORY

BACKGROUND

Competitive situations are distinguished by the fact that two or more individuals are making decisions in situations involving conflicting interests and in which the outcome is controlled by the decisions of all concerned. Such situations are found in many social, political, economic and military problems. Most of these situations involve elements of chance but in each case it may be assumed that each opponent will act in a rational manner and will attempt to resolve the conflict in his favor.

Von Neumann developed an approach to competitive problems utilizing the minimax principle which has as its basic idea the minimization of maximum loss. Von Neumann's ideas have led to the development of a branch of mathematics known as game theory in which the aim is to find the optimum strategy for a competition.

The foundations of game theory were laid in the book, "Theory of Games and Economic Behavior" by Von Neumann and Morgenstern[31], first published in 1944. Newman[32], in comenting on the social application of mathematics, wrote, - "the theory of games can fairly be said to have laid the foundation for the systematic and penetrating mathematical treatment of a vast range of problems in social science."

However, a more recent survey of game theory by Luce and Raiffa[33] indicates that there have been few applications in the social sciences. The majority of applications have been in military operations and in competitive industrial problems such as bidding strategies.

During its short history, game theory has had a number of contributors. Among the more prominent are George W. Brown who developed a method of finding solutions by ficticious play, and J. D. Williams who defined a simple method of interpereting equations called "the method of oddments". H. Kuhn developed the "kernel method" for solution of finite games and George Dantzig proved game problems could be converted to linear programming problems and vice versa.

New mathematical tools are being created to develop and extend the applications of game theory in the social sciences, operations research, statistics and other fields.

EVALUATION OF BIBLIOGRAPHY

For those seeking references and articles in specific areas of game theory, the excellent bibliography by Batchelor[34] is recommended as well as the Index of Publications of the Rand Corporation[35]. The Rand Corporation has sponsored the work of a number of men in this area. Both sources are well indexed.

Among texts dealing with the subject, Introduction to Operations Research by Churchman, Ackoff and Arnoff[36] is

good and includes the most complete treatment of competitive bidding models encountered in this study. The book, Mathematical Methods of Operations Research by Saaty[37], is also recommended.

A very different approach to game theory is found in Executive Decisions and Operations Research by Miller and Starr[38]. The treatment is rooted in decision theory and a general, non-mathematical, presentation is given. One of the better known works in game theory is the Compleat Strategist by Williams[39] which is suggested to those with a limited background in mathematics.

A variety of articles on applications and extensions of game theory can be found in the pages of the Journal of the Operations Research Society of America[40] and in Management Science.[41]

For those interested in applications of game theory in the area of war games, the paper by Dresher[42], covers some interesting aspects of target selection and prediction as well as optimal military strategy and tactics.

MATHEMATICAL METHODS

In the theory of games, we are dealing with systems in which two or more decision makers are in a competitive situation and in which the outcome is controlled by the decisions of all players. While not all competitive situations can be analyzed by means of game theory, there are many military, social, and economic situations which are

applicable. In line with the subject of this thesis, our purpose in examining game theory is to set forth methods for optimizing the strategy of a player.

A competitive game has the following four properties:

a. The number of players is finite.

b. Each of the N players has a finite group of possible courses of action which need not be the same for each player.

c. A play in the game occurs when each competitor selects one of his alternatives. It is assumed that selections are made simultaneously and without knowledge of the choices of other players.

d. The outcome of a play determines a set of payoffs (positive, negative, or zero), one for each player.

In a game where there are N players, and player i has n_i possible courses of action, there are $n_1, n_2, \ldots n_N$ possible outcomes for each play. An outcome θ results in a payoff $R(i,\theta)$ to player i. If for every possible outcome θ, we have

$$\sum_{i=1}^{i=n} R(i,\theta) = 0 \qquad (2\text{-}1)$$

then the game is called a zero-sum game. That is, if the sum of all payoffs to all players is zero, positive to some and negative to others, it is a zero-sum game. Poker is such a game.

The strategy of a player is the decision rule he uses for selecting a course of action. The decision rule should not require definite information about an opponents choice. A pure strategy is a decision by a player in advance of all plays, to select a single course of action which he then follows for all plays regardless of the outcome. A mixed strategy is a decision, in advance of all plays, to select an alternative for each play in accordance with some particular probability distribution. A mixed strategy is more advantageous than a pure strategy after the pattern of play becomes evident in that players are unable to predict the next play of their opponents.

A pure strategy is frequently identified by a single number such as 1 or 2 representing the course of action selected from all available alternatives. A mixed strategy, for a player with m possible courses of action, is denoted by the set, X, of m non-negative numbers whose sum is unity, representing the probabilities with which each alternative is chosen. If x_i is the probability of choosing course i, then the set X is $(x_1, x_2 \ldots\ldots, x_m)$

$$\text{where } x_i \geqq 0 \quad (i = 1, 2, \ldots\ldots, m)$$

$$\text{and} \sum_{i=1}^{i=m} x_i = 1 \tag{2-2}$$

It will be recognized that a pure strategy is a special case of a mixed strategy where all of the x_i are zero except one which has a value of 1. A player may select any one of

m pure strategies but he has an infinite number of possible mixed strategies.

Our purpose here is to select the optimal strategy for a player and for this we must have a criterion of optimality. The criterion normally used is known as the maximum criterion. This can be explained in that every player must be regarded as desiring to maximize his returns and minimize his losses. These desires are in conflict with those of other players since some player must lose if another is to win. The solution then consists of specifying the strategies for each player which will maximize his return as a winner and minimize his loss if he is a loser.

Two Person Zero-Sum Games

A game with only two players A and B is most conveniently described by means of a pair of matricies. Row designations for each matrix are the courses of action available to A; column designations are the courses available to B; cell entries are the corresponding payments to A for one matrix and to B for the other matrix . In a zero-sum two person game, the cell entry in B's payoff matrix is the negative of the corresponding entry in A's payoff matrix, and for this reason problems are usually worked out using only one matrix.

All two person zero-sum games can be solved. The simplest type of game is one where the stable optimal strategies are pure strategies. This is the case only if the payoff

matrix contains a saddle point. The saddle point is the element of a matrix that is both the lowest element in its row and the highest element in its column. In other words, a saddle point is at once the largest of the row minima and the smallest of the column maxima.

In such instances, the solution is for player A to follow a pure strategy corresponding to the row through the saddle point and for B to use his pure strategy corresponding to the column through the saddle point. The value of the game to a player is his expected gain in one play of the game when both players use their optimal strategies.

For purposes of illustration, a payoff matrix is given below in which the row minima have been circled and the column maxima have been enclosed in boxes.

				B		
		I	II	III	IV	V
	I	$\boxed{9}$	3	1	$\boxed{8}$	0
	II	6	5	$\boxed{4}$	6	7
A	III	2	4	3	3	$\boxed{8}$
	IV	5	$\boxed{6}$	2	2	1

It can be seen that a saddle point of 4 exists in this case. The optimal strategies of players A and B are thus II and III respectively. The value of the game to A is 4 and to B it is -4. When B learns that A is playing strategy II, he will follow strategy III because it minimizes his loss.

Mixed Strategies

When there is no saddle point, the best strategies are mixed strategies and the solution to the game consists of evaluating the probabilities with which each pure strategy should be used. If a mixed strategy is utilized a minimum gain cannot be assured on any specific play, however, we attempt to insure that the minimum possible value of the expected gain will be as large as possible.

Let us assume a competive two person zero-sum game with a payoff matrix as shown below.

	B		
	I	II	III
I	-1	2	1
II	1	-2	2
III	3	4	-3

It will be noted that there is no saddle point. Searching for pure strategies, A finds I to be most advantageous and B selects III, assuring themselves of gains of at least -1 and -2 respectively. (It will be recalled that the payoff to B is the negative of that in A's matrix.) If the players start with their pure strategies, A soon discovers that B always chooses III and realizes that he, A, can increase his gain from 1 to 2 by playing II. B then shifts to II to increase his gain to 2 on each play. The strategy for each

thus becomes one of continuous change. It is also true that, if one player ever realizes that the other is using a fixed pattern or sequence of strategies, he will be able to anticipate the next play of his opponent. A good way to avoid a fixed pattern of play is to make the strategy for each play a random variable.

Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ indicate the optimum mixed strategies for A and B respectively. The probability that A will play i is x_i and the probability that B will play j is y_i. The gain to A at each play now becomes a random variable α and the expected payoff of a play to A is:

$$E(\alpha;X,Y) = \sum_{ij} a_{ij}x_i y_j \tag{2-3}$$

where a_{ij} is the gain to A when A plays i and B plays j.

Without regard to Y, A seeks to choose X so that his expected gain exceeds some quantity v_1. The expected payoff for B similarly becomes:

$$E(-\alpha;XY) = \sum_{ij} (-a_{ij})x_i y_j \tag{2-4}$$

B plays so that his expected gain exceeds a quantity v_2. Designating optimum strategies as X_o and Y_o for A and B respectively, we must then satisfy the following conditions:

$$E(\alpha;X_o,Y) \geqq v_1 \text{ for all } Y \tag{2-5}$$

$$E(-\alpha;X,Y_o) \geqq v_2 \text{ for all } X \tag{2-6}$$

One of the theorems developed by Von Neumann states that in any zero-sum two person game, the quantities v_1 and v_2 above, are the negatives of each other. Substituting v for v_1 and -v for v_2, the following relations are obtained. Note that in equation (2-8), reversing the inequality sign causes the negative signs to be dropped.

$$E(\alpha;X_o,Y) \geqq v \text{ for all } Y \tag{2-7}$$

$$E(\alpha;X_1Y_o) \leqq v \text{ for all } X \tag{2-8}$$

Therefore:

$$E(\alpha;X_o,Y_o) = v \tag{2-9}$$

The interpretation of this is that if both players use their optimal minimax strategies, their actual expected gains coincide with their minimax expected gains.

Applying the foregoing to the numerical example on page   and substituting the values of a_{ij} in (2-7), we may obtain an expression for the expected gain to A for this game.

$$\begin{aligned} y_1(-x_1 + x_2 + 3x_3) + y_2(2x_1 - 2x_2 + 4x_3) \\ + y_3(x_1 + 2x_2 - 3x_3) \geqq v \end{aligned} \tag{2-10}$$

Since the foregoing expression is true for all y_j, it must be true when any two of the y_i are zero and the third is one. This observation allows us to create from the relation in (2-10), three inequalities which are:

$$-x_1 + x_2 + 3x_3 \geqq v$$
$$2x_1 - 2x_2 + 4x_3 \geqq v \qquad (2\text{-}11)$$
$$x_1 + 2x_2 - 3x_3 \geqq v$$

The first of these was obtained by setting $y_1 = y_2 = 0$; $y_3 = 1$. The second by setting $y_1 = y_3 = 0$; $y_2 = 1$ and the third by setting $y_1 = 1$; $y_2 = y_3 = 0$.

By a process similar to that used to obtain (2-10), we may write an expression for the expected gain to B and use this to write three more inequalities similar to those in (2-11) above.

$$-y_1 + 2y_2 + y_3 \leqq v$$
$$y_1 - 2y_2 + 2y_3 \leqq v \qquad (2\text{-}12)$$
$$3y_1 + 4y_2 - 3y_3 \leqq v$$

It is also true that:

$$x_i \geqq 0 \qquad (i = 1, 2, 3)$$
$$\qquad\qquad (2\text{-}13)$$
$$y_i \geqq 0 \qquad (i = 1, 2, 3)$$

Since the sum of the probabilities x_i, is equal to 1 from (2-2), we can also write:

$$x_1 + x_2 + x_3 = 1$$
$$\qquad\qquad (2\text{-}14)$$
$$y_1 + y_2 + y_3 = 1$$

The equations and inequations included in (2-11) through (2-14), describe the seven unknowns x_i, y_j and v. It is

apparent that not all the x_i and y_i can be zero, but it is possible for equality signs to hold in all the relationships of (2-11) and (2-12). This is first assumed to be the case and we can undertake a solution of the following set of eight equations with seven unknowns:

$$\begin{aligned}
-x_1 + x_2 + 3x_3 &= v \\
2x_1 - 2x_2 + 4x_3 &= v \\
x_1 + 2x_2 - 3x_3 &= v \\
-y_1 + 2y_2 + y_3 &= v \\
y_1 - 2y_2 + 2y_3 &= v \\
3y_1 + 4y_2 - 3y_3 &= v \\
x_1 + x_2 + x_3 &= 1 \\
y_1 + y_2 + y_3 &= 1
\end{aligned} \qquad (2\text{-}15)$$

If positive solutions are found for the x_i and y_i, then all relations are satisfied. If not, one or more of the equality signs in (2-15) must be replaced by inequalities until positive solutions are found.

In this instance, the equations hold and positive solutions are:

x_1	x_2	x_3	y_1	y_2	y_3	v
17/46	20/46	9/46	14/46	12/46	20/46	30/46

Note that A now has an expected gain per play of 30/46, where with his best pure strategy he could only be sure of not losing more than 1. B now has an expected loss of

-30/46, where with a pure strategy he could not be sure of losing less than 2.

Solution of Rectangular Games

Finite two-person zero-sum games are referred to as rectangular games. Three theorems due to Von Neumann are basic to the solution of rectangular games and will be reviewed here in brief.

Theorem 1. Every rectangular game has a value and a player of a rectangular game always has an optimal strategy.

Theorem 2. v^*, X^*, and Y^* are respectively the value of a rectangular game and optimal strategies for players P_1 and P_2, if for pure strategies the following relations are true:

$$E(X_p, Y^*) \leqq v^* \text{ for every pure strategy } X_p \text{ of } P_1 \quad (2\text{-}16)$$

$$E(X^*, Y_p) \geqq v^* \text{ for every pure strategy } Y_p \text{ of } P_2 \quad (2\text{-}17)$$

Theorem 1 indicates that each rectangular game has a solution with a value equal to some finite number. The theorem does not say that each game has a unique solution and there may be more than one optimal strategy.

Theorem 2 serves to check proposed solutions and if the relations do not hold for the proposed solution, it is not valid.

With this background, let us examine briefly a few of the methods of solution which have been developed for rectangular games.

2 X 2 Games

The general payoff matrix for a 2 X 2 game is shown here:

		P_2	
	Alternative	B_1	B_2
P_1	A_1	a_{11}	a_{12}
	A_2	a_{21}	a_{22}

If the pay-off matrix has a saddle point, the solution is direct and need not be considered further. However for purposes of discussion let us assume that the game has no saddle point. For this situation it may be shown that the specific relationships among the elements a_{ij} are:

$$
\begin{array}{ll}
a_{11} > a_{12} & a_{11} > a_{21} \\
a_{22} > a_{12} & a_{22} > a_{21}
\end{array}
\qquad (2\text{-}18)
$$

If these relations were not true, a saddle point would exist.

Let the general solution of this game be

v_o, $X_o = (x_o, 1-x_o)$, $Y_o = (y_o, 1-y_o)$

$$
\begin{array}{ll}
\text{where} & 0 \leqq x_o \leqq 1 \\
\text{and} & 0 \leqq y_o \leqq 1
\end{array}
\qquad (2\text{-}19)
$$

By theorem 2:

$$
\begin{aligned}
a_{11}y_o + a_{12}\,(1-y_o) &\leqq v_o \quad \text{for } A_1\\
a_{21}y_o + a_{22}\,(1-y_o) &\leqq v_o \quad \text{for } A_2\\
a_{11}x_o + a_{21}\,(1-x_o) &\geqq v_o \quad \text{for } B_1\\
a_{12}x_o + a_{22}\,(1-x_o) &\geqq v_o \quad \text{for } B_2
\end{aligned}
\tag{2-20}
$$

It has been demonstrated[43] that these four inequalities have a solution only when the equalities hold, the solution being:

$$v_o = [(a_{11}a_{22} - a_{12}a_{21})]/[(a_{11} + a_{22}) - (a_{12} + a_{21})] \tag{2-21}$$

$$x_o = (a_{22} - a_{21})/[(a_{11} + a_{22}) - (a_{12} + a_{21})] \tag{2-22}$$

$$y_o = (a_{22} - a_{12})/[(a_{11} + a_{22}) - (a_{12} + a_{21})] \tag{2-23}$$

This means that players P_1 and P_2 should use the following optimal mixed strategies:

$$X_o = \left[\frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \right. \tag{2-24}$$

$$\left. \frac{a_{11} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}\right] \text{ for } P_1$$

$$Y_o = \left[\frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \quad \frac{a_{11} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}\right] \text{ for } P_2 \qquad (2\text{-}25)$$

A method for interpreting these two equations was presented by J. D. Williams[44], which he terms the method of oddments. Williams calls the difference between two numbers in a row or column an oddment. He terms the ratio of the two relative frequencies of any player, odds. The optimal mixed strategy for P_1 thus is to play A_1 and A_2 with the odds of oddment $a_{22} - a_{21}$ to oddment $a_{11} - a_{12}$. The optimal mixed strategy for player P_2 is to play B_1 and B_2 with odds of oddment $a_{22} - a_{12}$ to oddment $a_{11} - a_{21}$. In more familiar numerical terms, if the payoff matrix is such that:

$$a_{22} - a_{21} = 3$$

$$a_{11} - a_{12} = 5$$

Then the optimal odds for player P_1 are 3 to 5 and he should thus play alternate A_1 3/8 of the time and A_2 5/8 of the time.

M X N Games

Several methods have been developed for solution of games of this type. All are relatively long and no attempt will be made to cover them in detail here. A few will be briefly described.

In selecting an approach to a game and before attempting a solution, two steps should be taken. As stated earlier, the first thing to do is to examine the problem for a saddle point. Failing this, certain alternatives can be eliminated by checking for what is termed dominance. When the payoffs for alternative A_1 are greater or equal to each of the corresponding payoffs for A_2, A_1 is then said to dominate and A_2 may be disregarded. The concept of dominance may be extended to more than two alternatives at a time. McKinsey[45] and Von Neumann[46] give detailed examples of this approach.

Ficticious Play Solution

George W. Brown's[47] method of solution by ficticious play is based on a hypothetical series of consecutive plays of the game and uses an infinite sequence of steps to reach a solution. In the first play, P_1 selects A_1 and P_2 plays B_1. Thereafter each player chooses the optimum pure strategy against the mixture represented by all of his opponents past plays. Saaty[48] presents a good example of this method. There are other techniques of solution including the geometric, algebraic, matrix and Kernel methods. For examples and for further information on these methods, the reader is referred to Saaty[49] or McKinsey[50].

Linear Programming Solution

It has been shown by Dantzig[51] that every game problem can be transformed into a linear programming problem and vice versa. Saaty[52] presents some examples of how this is

accomplished and it is sufficient for us to note that Dantzig has proven the validity of this transformation and it becomes a valuable tool in both the solution of game and linear programming problems.

N Person Zero-Sum Games

It should be noted that we have been dealing with relatively simple games involving two players. No generally acceptable definition of a solution of an n person zero-sum game yet exists. One reason for the lack of an acceptable definition is that coalitions can form when more than two players participate. Von Neumann and Morgenstern[53] have suggested an approach based on possible formation of coalitions involving dominance. However their definition of a solution does not show how to find optimal strategies nor indicate what the final outcome will be. For these reasons Von Neumanns definition of n-person games has not had wide acceptance.

Non-Zero Sum Games

A discussion of the methods of game theory would not be complete without some mention of non-zero sum games. Here too, there is no generally acceptable definition of a solution. Von Neumann[54] suggests an approach introducing a ficticious player assigning him such payoffs that the game is reduced to a zero-sum game. The solution is then attempted by methods applicable for zero-sum games. However, there are infinitely many solutions and no acceptable definition has been found.

APPLICATION OF GAME THEORY

The most extensive applications of game theory have been in the area of military warfare. It has also been applied to economic problems involving commercial competition and in both cases the complexities of the problem must be simplified by various assumptions to make it amenable to analysis. This creates some doubt as to the validity of the results, especially where the game gives nearly equal opportunity to the players. In the case of war games, frequently one side has a considerable advantage and the use of game theory analysis will usually reveal this advantage which might otherwise remain obscured.

In the area of military petroleum problems, the petroleum logistics involved would usually be a portion of the battle game which will not be treated here. There is, however, an application of game theory in the area of competitive bidding which has a direct bearing on military petroleum procurement which will be presented here as an example. While this competitive situation differs from the strict sense of a competitive game, it is possible to use similar means of analysis.

Competitive Bidding

As a simple case, let us assume a refiner engaged in bidding on military petroleum contracts, makes it a practise to record, in addition to his own bid, his cost estimate K

and the lowest bid B by his opponent. The refiner finds that the random variable x, where

$$x = \frac{B - K}{K},$$

follows a normal distribution f(x) with a mean of 0.15 and standard deviation of 0.05.

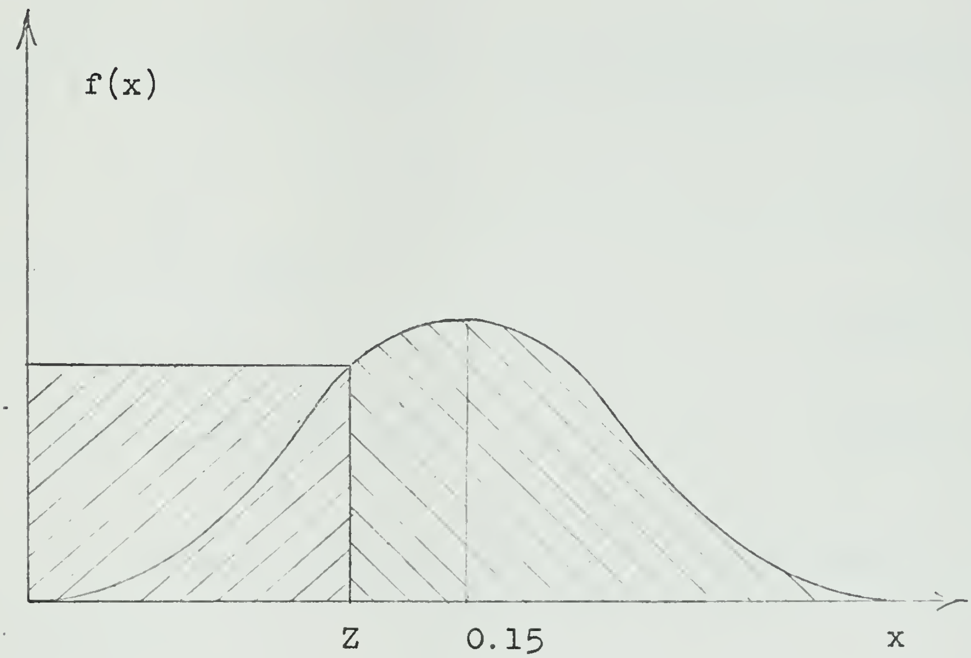

Figure 5

Graphical Solution of Equation (2-32)

If the refiner is interested in maximizing his expected profit on each contract, and if it is assumed that his competitors bidding habits will not change, what should the refiners bidding strategy be?

Let A be the refiners' bid and B be the lowest competing bid. The refiners' profit P is thus:

$$\begin{aligned} P &= A - K \quad \text{if} \quad A < B \\ P &= 0 \qquad\;\;\; \text{if} \quad A > B \end{aligned} \tag{2-26}$$

Let E(P;A) signify the expected profit with bid A. Then:

$$E(P;A) = (A - K)P \quad \text{if} \quad A < B \tag{2-27}$$

Replacing the decision variable A by an equivalent variable Z, where

$$Z = \frac{A - K}{K} \tag{2-28}$$

Substituting from (2-28)

$$E(P;A) = KZP \quad \text{if} \quad Z < x \tag{2-29}$$

$$E(P;A) = KZ \int_{Z}^{\infty} f(x)\, dx \tag{2-30}$$

Where x is the random variable and f(x) is its function.

The quantity

$$Z \int_{Z}^{\infty} f(x)\, dx \tag{2-31}$$

is maximized by the solution Z of the equation

$$\int_{Z}^{\infty} f(x)\, dx - Z\, f(Z) = 0 \tag{2-32}$$

Graphically, E(P;Z) is a maximum when the two cross hatched areas in Figure 5 are equal.

5

Numerically, Z may be found by using tables of ordinates and areas of normal distribution. In this instance, $Z = 0.117$ is a solution to the foregoing equation. Thus the refiners optimum bidding strategy is to add 11.7% to his own cost K, in preparing his bid.

A critical assumption is that the competitors will not change their bidding habits. In the case where only two bidders are in competition, such an assumption will not hold, however, it is nore likely that where several competitors exist, it may be more reasonable to assume that the bidding pattern will not change.

In the case of petroleum refiners, competing companies have a fair amount of information on their opponents capabilities and markets plus a reasonably good estimate of their costs. This information enables them to bid more intelligently, no doubt, than the refiner in the example. It is known that petroleum refiners in at least a few cases, have prepared programs for computer calculation of bids. It is not surprising that these computer programs are not available for analysis.

CHAPTER III

INVENTORY THEORY

BACKGROUND

Probably more work has been directed toward inventory control than toward any other problem area in business, industry, and government. In spite of this, or perhaps because of it, a problem in terminology exists in the area. Moreover, there are widely divergent views on what research in this area should include. Some references treat inventory systems as record-keeping problems involving quantities of stock while others take a broader view and consider the overall problem of stocks and inventory investment. Still others are concerned with what items to stock, quantities, and utilization of facilities. The diversity stems from the many facets of what has been called the inventory system and the large variety of systems found in practise.

In this thesis the basic characteristics of inventory systems, input, inventory levels, and output as well as the costs associated with these factors, will be discussed and and applied in a few models. The aim in the handling of each inventory problem is one of optimizing costs within the limits of effectiveness required by management policies.

Historically, one of the earliest efforts at analysis of inventory systems was made by F. W. Harris in 1915. Harris recognized many of the elements of inventory problems

and developed a formula for the determination of optimal lot size.[55]

$$q_o = \sqrt{2rc_3/c_1} \qquad (3\text{-}1)$$

In this formula q_o is the economic lot size, r is the rate of requirements per unit time, c_1 is the carrying cost of one unit in inventory per unit time, and c_3 is the cost of replacing inventory. Application of this formula has been extensive; however, Harris assumed that the rate of requirements were known, that no shortages occurred, and that lead time could be disregarded. These limitations made it useful only in special cases and improvements were sought.

Cooper in 1926 and Fry in 1928 analyzed systems with varying production and demand. A significant advance was made by Fry when he showed that the theory of probability can be applied to some inventory systems when requirements are not known with certainty.

Another important step was taken by Raymond who made the first effort to analyze and develop a theory of inventory systems in his book Quantity and Economy in Manufacture published in 1931 and no longer in print.[56]

Dvoretsky, Kiefer and Wolfowitz[57] published a paper, "The Inventory Problem", in 1952 in which they studied systems of a general nature utilizing extensive mathematical and statistical methods and endeavored to provide solutions for general cases. While they were successful in obtaining some unique solutions, there is some danger in

applying general solutions to the analysis of specific inventory systems and their work is chiefly of theoretical interest.

The approach taken by Arrow, Harris, and Marschak[58], as well as Whitin[59], was to analyze specific hypothetical systems encountered in practise and apply the explicit rules developed to actual cases.

A different approach by Naddor[60] and others, places emphasis on the development of methods of analysis. Their intent is to develop models of elementary systems and analyze more complicated systems as extensions of the basic models. In reviewing the numerous sources of information on inventory theory, the text by Churchman, Ackoff, and Arnoff[61] is considered the best for those seeking basic knowledge on inventory and purchasing. Another recommended source is Production Planning and Inventory Control by Magee[62].

A review of the references and bibliography at the end of this thesis will suggest additional sources of information. The periodical, Management Science[63], is an excellent source of information on inventory theory and other related subjects.

ANALYSIS OF INVENTORY SYSTEMS

An inventory system is usually considered to be one in which the following types of costs are significant, and in which any two or all three are subject to control:

1. the cost of carrying inventories,
2. the cost of incurring shortages,
3. the cost of replenishing inventories.

The first cost relates to inventory investment, storage and handling costs, and losses due to spoilage, evaporation, etc. The second, shortage cost, concerns the cost due to lost sales, overtime, special handling, etc. The third cost is that of ordering, machine set up, etc. The general terms for the three costs are surplus cost, shortage cost and set-up cost. As will be noted in some applications involving use of inventory theory, all other costs become significant in actual cases and problems do not limit themselves to the area of inventories alone. Actual problems frequently require use of other optimization techniques in order to avoid what we have termed sub-optimization.

Inventory Problems

An inventory problem is one of making optimal decisions with respect to an inventory system in order to minimize the total cost of the system. The measure of effectiveness in inventory problems is one of costs. It is assumed that all shortages, surpluses and set-ups can be expressed in common units of the measure of effectiveness even though the units may not be dollars.

Control of Variables

While it has been indicated that the surplus, shortage and set-up costs are the factors which we desire to control,

the optimal decisions designed to minimize total costs are normally made in terms of time and quantity. More specifically, the decision is frequently one of when to replenish the inventory and by how much. The variables thus subject to control are usually time and quantity.

Elements of Inventory Systems

The three elements of inventory systems are output, input, and costs. Output refers to what is removed or issued from inventory. Input includes what is received or taken into inventory and costs refers to the three costs mentioned earlier (surplus, shortage and set-up costs).

Output: In several respects, the output of a system is the most important element. Inventories are maintained to fill demands. The demand or output cannot usually be controlled but the pattern of output in terms of time and quantity has a significant bearing on the operation of the system. The output may be variable or constant. When accurate advance information about the size of the output is available, it is said to be known with certainty. When not known it may be subject to a known probability distribution. The output rate or demand rate is the output size per unit time and it may be constant or vary in a manner similar to the output size.

Output patterns vary widely but may generally be described mathematically if sufficient data are available. If the output size is x during an interval of time t, the output

pattern may follow one of the patterns in Figure 6. All x units may be drawn at the beginning or at the end of the period, uniformly during the period, or in accordance with some other distribution.

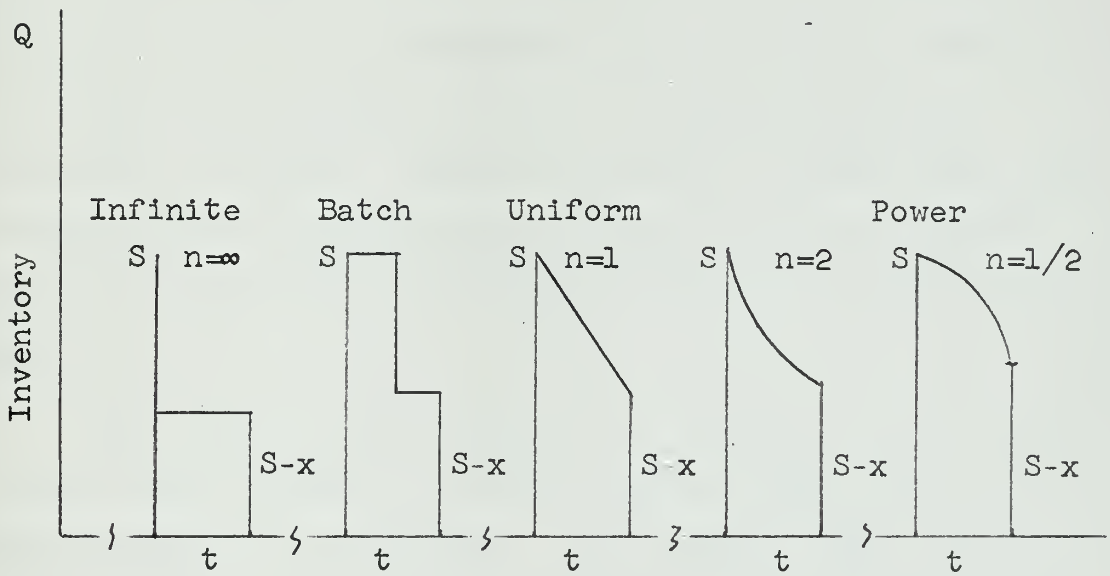

Figure 6

Output Patterns

The output patterns shown can be described by:

$$Q(T) = S - x\sqrt[n]{T/t} \qquad (3\text{-}2)$$

where $Q(T)$ is the total inventory at time T

S is the inventory when T = 0

x is the output during the interval t

n is the output pattern index

t is the scheduling period

<u>Input</u>: The input element can generally be controlled. Input involves not only the quantity scheduled to be taken into inventory, but the time of ordering or scheduling as well as the time of receipt into stock. The scheduling period is the term used for the period between decisions regarding inputs. This cannot always be controlled. Lead time is the period between scheduling or ordering and receipt and this too is frequently not subject to control. The input size is defined as the quantity ordered or scheduled for input. Numerous inventory problems are concerned with the optimization of the input size in which case it is frequently called the optimal or economic lot size.

Input patterns are related to the input period which is the time interval t' during which the input size q is being added to the inventory. The average input rate p is the ratio of input size to input period or units of input per unit time.

$$p = q/t' \tag{3-3}$$

Input patterns vary widely as illustrated in Figure 7 and each type is important in some particular problem.

All of the patterns shown have an input size q being added at average input rate p. This group of patterns can be described by:

$$Q(T) = q\sqrt[n]{T/t'} = q\sqrt[n]{Tp/q} \tag{3-4}$$

where: Q(T) Inventory at time T

q Input size

t' Input period

p Average input rate (p = q/t')

n Input pattern index

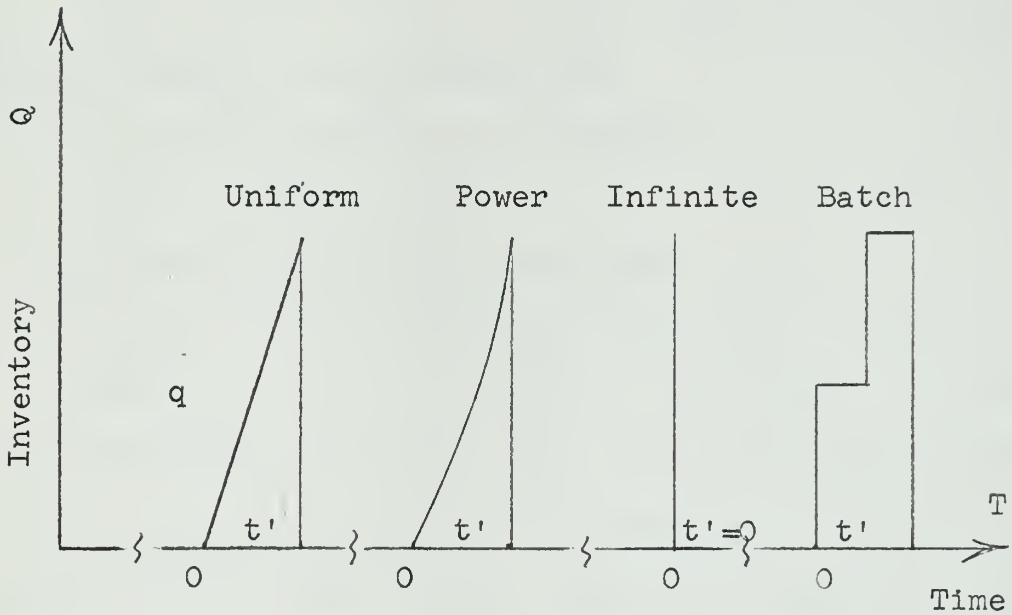

Figure 7

Input Patterns

<u>Costs</u>: The cost element of inventory systems is considered to have been covered sufficiently in the section headed Analysis of Inventory Systems on page 63. If any further clarification of the types of costs included under the titles of surplus costs, shortage costs, and set-up costs is desired, the following references are recommended: 64, 65.

APPLICATIONS OF INVENTORY THEORY

In order to illustrate the application of inventory models, a few types will be presented here, largely in the form of examples. Among basic inventory systems, there are three types all of which involve the balancing of two of the three types of costs; surplus, shortage and set-up.

Balancing Surplus and Shortage Costs

The following conditions are known:

c_3 (set-up cost) = constant

t_p (scheduling period) = constant

r (output ratio in units per unit time) = constant

c_1 (surplus cost of one unit per unit time) = constant

c_2 (shortage cost of one unit per unit time) = constant

The inventory level S at the beginning of each scheduling period is the only variable subject to control in this model. The system may be illustrated graphically as shown in Figure 8.

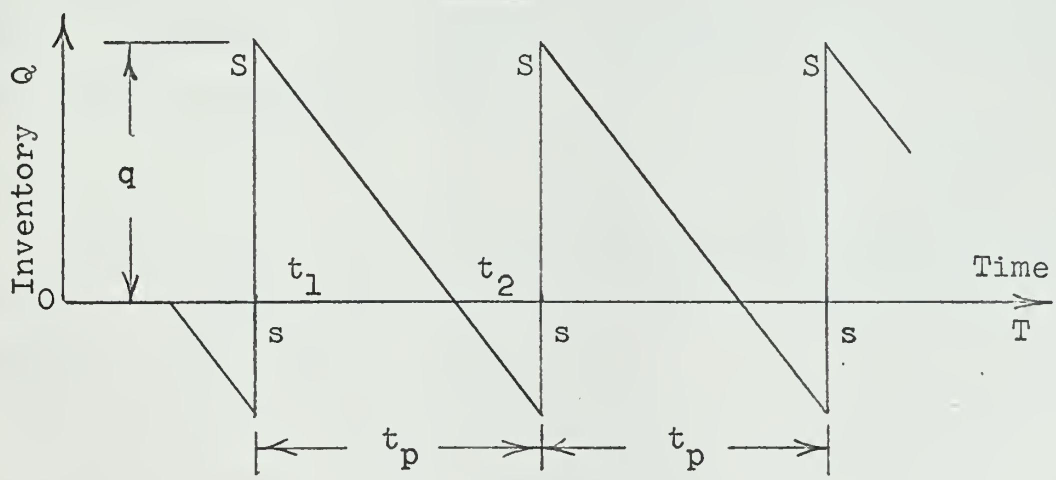

Figure 8

System with Surplus and Shortage

The sawtooth line in the figure represents the inventory level at various times. The initial inventory level S is drawn down at a constant rate r. At time t_1, the inventory reaches zero and a shortage s occurs. The demands which represent shortages are accumulated and stock is issued to cover these demands after the inventory is replenished. Input q is sufficient to cover the shortage and raise the inventory level again to S. Note that the vertical input line indicates practically instantaneous receipt of stock at each scheduling period t_p.

Since shortages are carried over as indicated, the input q becomes:

$$q = rt_p \qquad (3\text{-}4)$$

where r is the rate of output and t_p is the scheduling period or time between inputs.

The inventory level, s, which is actually the amount of shortage at the end of the scheduling period, is dependent upon the variable S as well as on the other conditions given. Although s is a negative inventory, it is treated as a positive quantity here:

$$s = rt_p - S \qquad (3\text{-}5)$$

The solution of this system requires the determination of S_o, the optimal inventory level, and C_o, the optimal (minimum) total cost per unit time.

The costs of the system are defined by the equations:

$$\text{Surplus cost per unit time } C_1 = I_1 c_1 \qquad (3\text{-}6)$$

Shortage cost per unit time $C_2 = I_2c_1$ (3-7)

Setup cost per unit time $C_3 = Nc_3 = c_3/t$ (3-8)

where I_1 and I_2 respectively, are the average surplus and shortage during a unit of time and N is the number of setups per unit time.

Using equations (3-6), (3-7), and (3-8) as well as the relationships from Figure 8, one may derive the following equations which describe the surplus costs and shortage costs of the system:

$$C_1 = (1/2)S(t_1/t_p)c_1 = S^2c_1/(2rt_p) \tag{3-9}$$

$$C_2 = (1/2)s(t_2/t_p)c_2 = s^2c_2/(2rt_p) \tag{3-10}$$

The derivation of (3-9) can be explained briefly by pointing out that the term (S/2) represents the average surplus inventory and that the term (t_1/t_p) represents that proportion of the scheduling period during which a surplus exists. The product of these terms multiplied by c_1, the surplus cost of one unit per unit time, thus equals the total cost of surplus C_1. The alternate form of the equation is obtained by substituting the relation $t_1 = S/r$ which is the time required to deplete the initial inventory S at output rate r. Equation (3-10) is derived similarly.

Adding (3-9) and (3-10) one may obtain the total cost per unit time C which is subject to control since C_3, the setup cost, is constant.

$$C = C_1+C_2 = [S^2c_1 + (rt_p-S)^2c_2]/(2rt_p)$$

$$= [1/(2rt_p)](c_1+c_2)S^2 - c_2S + c_2rt_p/2 \tag{3-11}$$

This equation, usually called the total cost equation, can be used to arrive at the optimal decision rules by differentiating, setting the derivative dC/dS equal to zero, and solving for S_o, the optimal inventory level. The optimal inventory level then becomes:

$$S_o = rt_p\, c_2/(c_1+c_2) \tag{3-12}$$

This value of S can be substituted in equation (3-11) to obtain the minimum total cost, c_o.

$$C_o = (1/2)(rt_p c_1 c_2)/(c_1+c_2) \tag{3-13}$$

Pump Seal Problem

The foregoing model will be applied here in an example to illustrate the use of the equations.

A fuel terminal uses 2400 pump seals annually due to rapid wear. When a seal is needed but not available, leakage occurs costing $9 per month. 200 seals are ordered each month and delivered in one day. Each seal costs $80. Inventory carrying cost is 15% per year.

In terms of the designations given:

(Scheduling period) t_p = 1/12 year

(Output rate) r = 2400 per year

(Unit surplus cost) c_1 = $80 x 0.15 = $12 per seal per year.

(Unit shortage cost) c_2 = $9 per part per month = $108 per seal per year

From equation (3-11)

$$C = 0.3\ S^2 - 108\ S + 10,800$$

If no shortages were permitted, S would have to equal 200 seals in which case the total cost would be $1200 per year. However, when shortages are allowed and surplus and shortage costs are balanced, the optimal inventory level, from equation (3-12), becomes; $S_o = 180$ seals. The minimum total cost for the year is then $1080 per year.

Balancing Surplus and Set-up Costs

An inventory system in which the cost and consequences of shortage is so high in terms of lost operating time, etc., that shortages are unacceptable, may still be optimized by balancing the surplus and setup costs. Such a system is undoubtedly appropriate for most packaged petroleum products for military use in that a shortage of a relatively inexpensive special purpose grease may prevent the operation of an aircraft or other vital equipment. In setting up this particular model, no reserve inventory level is included. This would not be the practise in an actual case but since a reserve level would simply constitute the addition of a constant cost, the treatment is no less applicable. An illustration of such a system is shown in Figure 9.

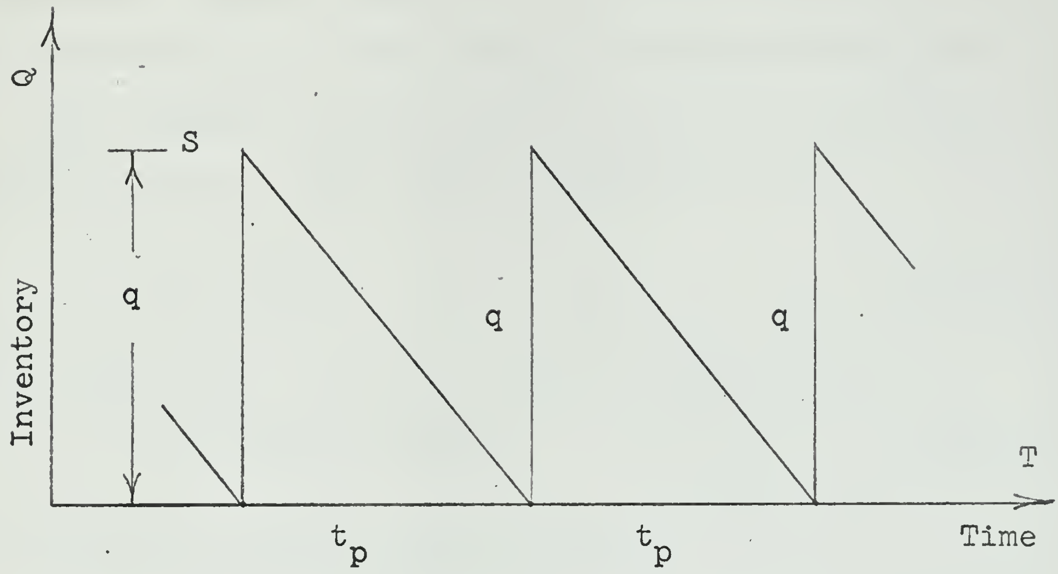

Figure 9

System with Surplus and Setup Cost

The conditions describing the system are:

(No shortages allowed) $I_2 = 0$

(Output rate in units per unit time) r = constant

(Cost of carrying one unit per unit time) c_1 = constant

(Cost of each setup) c_3 is known

The variables in this model are the input q and the scheduling period t_p. The relationship between the variables is:

$$q = rt_p \tag{3-14}$$

It is assumed that the variable subject to control is q the input quantity. In this system, since no reserve

exists, the initial inventory S is equal to q and the shortage carryover, s, is zero since no shortage is allowed.

The average inventory level during the scheduling period t_p is $I_1 = q/2$, thus by equation (3-6)

$$C_1 = qc_1/2 \tag{3-15}$$

and by equations (3-8) and (3-14) one may write:

$$C_3 = c_3/t_p = rc_3/q \tag{3-16}$$

The total cost equation then becomes:

$$C = C_1 + C_3 = qc_1/2 + rc_3/q \tag{3-17}$$

By differentiating and setting the derivative dC/dq equal to zero, we can solve for the optimal value of q which gives the minimum total cost C. The optimal input value, q, is thus:

$$q = \sqrt{2rc_3/c_1} \tag{3-18}$$

Equation (3-18) is frequently termed the "economic lot size" formula.

Entering q in (3-14) and (3-17) is then:

$$t_o = \sqrt{2c_3/(rc_1)} \tag{3-19}$$

$$C_o = \sqrt{2rc_1c_3} \tag{3-20}$$

The two equations (3-18) and (3-19), provide the optimal lot size and optimal scheduling period as a basis for

the decision rules in operating the system in which surplus and setup costs are balanced while no shortages are allowed. Equation (3-20) describes the minimum cost of such a system.

Pump Seal Problem (No shortage allowed)

The model of the foregoing system, in which no shortages were allowed, can be applied in the pump seal problem utilized in the previous application. In this instance we will assume that the scheduling period, t_p, is variable and not fixed; that no shortages are permitted; and that the terminal operator decides to make the seals with a setup cost of \$350 for each production run.

In this case it is necessary to decide how many seals should be made in each production run and how often this is to be done.

From the previous problem, it is given that:

Output rate r = 2400 seals per year

Surplus cost c_1 = \$12 per seal per year

Setup cost c_3 = \$350 per setup

Substituting in equations (3-18), (3-19), and (3-20):

Optimal lot size = q = 374 parts per run

Optimal scheduling period = t_o = 1.87 months

Optimal total cost = C_o = \$4,488

Application Involving Probabalistic Output

An inventory system with a probability distribution of output is illustrated graphically in Figure 10.

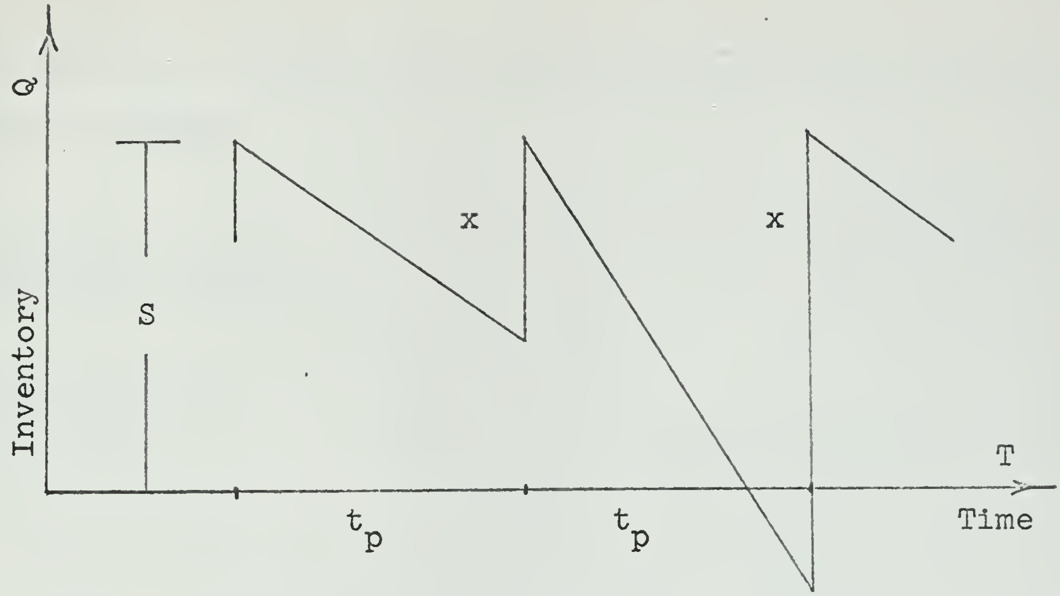

Figure 10

System with Probability Distribution of Output

Note that during each scheduling period, t_p, the system may either have an output size x which is smaller or equal to the initial inventory level S, or it may be that x is larger than S. For each of these two cases the distribution appears as shown in Figure 11.

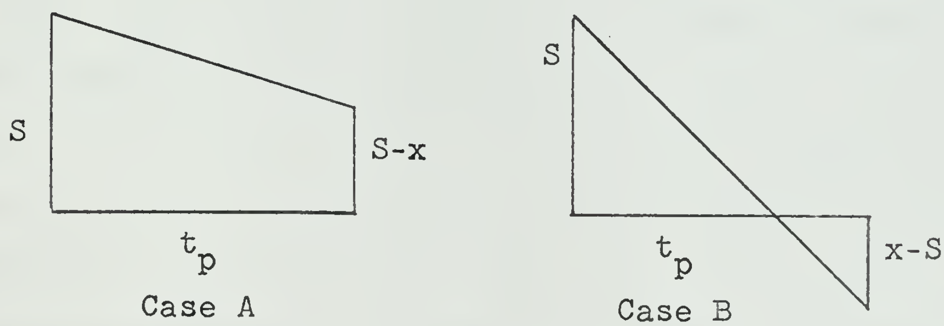

Figure 11

Typical Cases with Probability of Output

the pump seal problem, the following probability distribution is assumed:

Output size during Scheduling Period t_p	x	0	100	200	300	400	500
Probability of Occurrence	P(x)	.15	.20	.35	.15	.10	.05

Table 1

Probability Distribution of Output

In this problem it will also be assumed that the scheduling period, t_p, is fixed at one month. It is also given that:

Unit surplus cost c_1 = \$12 per unit per year

Unit shortage cost c_2 = \$108 per unit per year

Calculations are presented here in tabular form. The descriptions entered at the head of each column will assist in following the computation.

The computations were initiated by assuming various inventory levels, S, and values of output, x, varying from 0 to 500. One can then calculate the average inventory for cases where there is no shortage quite simply. For example, when S is 100 and x is 100, the inventory is drawn down to zero at the end of the scheduling period and the average inventory for the month is 100/2 = 50. When there is a shortage, as when S is 100 and x is 200, the inventory is drawn down to zero in the first half of the month. In this

Inventory Level	Output Size	Probability of Output	Average Surplus Case A	Average Surplus Case B	Average Shortage Case B	Surplus Cost	Shortage Cost	Total Expected Cost	Equation (3-21) Equation (3-22)	Equation (3-23)
S	x	P(x)	I_{1a}	I_{1b}	I_{2b}	C_1	C_2	$(C_1+C_2)[P(x)]$	C_a, C_b	C
100	0	.15	100			1200		180	300	
	100	.20	50			600		120		
	200	.35		25	25	300	2700	1050		
	300	.15		17	67	204	7236	1116		
	400	.10		12	112	144	12096	1224		
	500	.05		10	160	120	17280	870	4260	4560
200	0	.15	200			2400		360		
	100	.20	150			1800		360		
	200	.35	100			1200		420	1140	
	300	.15		67	17	804	1836	396		
	400	.10		50	50	600	5400	600		
	500	.05		40	90	480	9720	510	1506	2646
250	0	.15	250			3000		450		
	100	.20	200			2400		480		
	200	.35	150			1800		630	1560	
	300	.15		104	4	1248	432	252		
	400	.10		78	28	936	3024	396		
	500	.05		62	62	744	6696	372	1020	2580
300	0	.15	300			3600		540		
	100	.20	250			3000		600		
	200	.35	200			2400		840		
	300	.15	150			1800		270	2250	
	400	.10		112	12	1344	1296	264		
	500	.05		90	40	1080	4320	270	534	2784

Table 2

Solution of System with Probability Distribution of Output

One may assume that an output of size x will occur during t_p with a probability P(x). When $x \leq S$ the average surplus inventory is S-x/2 and the total surplus cost per unit time is $(S-x/2)c_1$. The total expected cost C_a for Case A (where all values of $x \leq S$) is thus:

$$C_A = c_1 \sum_{x=x_1}^{x=S} (S-x/2)\, P(x) \tag{3-21}$$

In Case B, when x > S, the total expected cost C_b from Equation (3-11) becomes:

$$C_b = \sum_{x=S+1}^{x=x_{max}} \left\{ [S^2 c_1 + (x-S)^2 c_2]/(2x) \right\} P(x) \tag{3-22}$$

Note that the output size $x = rt_p$. In this derivation x is substituted for (rt_p) in Equation (3-11).

The total cost equation then becomes:

$$C = C_a + C_b = c_1 \sum_{x=x}^{x=S} (S-x/2)P(x) + c_1 \sum_{x=S+1}^{x=x_{max}} (1/2)S^2[P(x)]$$

$$+ c_2 \sum_{x=S+1}^{x=x_{max}} (1/2)(x-S)^2[P(x)/x] \tag{3-23}$$

Pump Seal Problem with Probability Distribution of Output

In actual cases, the output of an inventory system is rarely constant but if sufficient data are available, a probability distribution for output size can be determined. In

case the average inventory is 100/2 = 50 for only one half of the month so for purposes of this computation, the actual average during the scheduling period of one month is 25. Columns C_1 and C_2 are obtained by multiplying the average surplus or shortage by the unit surplus or unit shortage costs as applicable. The remainder of the calculation is generally straight-forward.

In order to obtain the optimum inventory level S_o for this problem as well as the corresponding minimum cost, it is necessary to use interpolation and some trial and error calculations. In this instance, the approximate values of S_o can be seen to lie between 200 and 300 and comparatively close to 250. By additional computation it appears that S_o is approximately 235 seals at which inventory level the minimum total cost C_o is $2576 per year.

<u>Application of Inventory Theory in Military Tanker and Terminal Operations</u>

One of the important problems facing military and industrial petroleum management, is that of optimizing the operation of systems encompassing the movement, storage and distribution of bulk petroleum products. The large amount of capital and effort expended in the operation of such systems, makes them an important field for analysis.

The military services are concerned with systems involving the transportation of products by tanker from refineries to terminals for storage and distribution. The

petroleum industry operates similar systems in that refined products are transported by tanker to regional terminals for storage and distribution. The industry operates another such system in the tanker movement of crude oil from terminals to refineries for processing. As operated by industry, these two systems are, of course, related in that the output of the crude oil system becomes the input of the refined product system.

The similarity of the military and industrial refinery - tanker - terminal systems, suggests that optimization techniques applied by industry may prove valuable to the military services. It is the purpose and intent of this thesis to present such applications, and we will proceed on the assumption that recognition of the basic factors involved will assist toward a solution.

Inventory - Queueing Systems

Tanker-terminal systems are essentially inventory systems, however, the problems associated with the arrival of tankers for service at a refinery or terminal involves certain aspects of queueing theory. Since the subject of queueing has not been covered in this thesis, it is considered that a brief discussion of the elements of queueing situations and their solution by Monte Carlo methods should be included at this point as an aid to understanding what will follow.

Elements of Queueing Theory

Queueing or waiting line theory concerns the attempt to describe and handle problems of organization and planning in the face of randomly fluctuating demands for a servive being performed. Situations frequently occur in which units arriving for service must wait before being serviced. If the conditions controlling arrivals, service times, and the order of servicing are known, then the nature of the waiting line can be analyzed mathematically.

Two general classifications can be made, one a determinate type of operation and the other indeterminate. In the former all aspects of the system are known as functions of time and the state of the system can be predicted at any time. In the indeterminate case, aspects of the system are known only in terms of probabilities.

The several characteristics of a queue, such as the number in line at any instant or the waiting time experienced by a particular arrival, are random variables rather than being functionally dependent on time. Arrivals and service times are also considered to be random variables. Thus in determining the values of the characteristics of a queue, one is concerned with estimating only the average number of units in the line at any instant, the average time the service facility is idle during a day, the average service time and the average waiting time.

In attempting to optimize a queueing system, the decision maker uses his knowledge of the average characteristics

of the queue. He may consider: altering the number of service units, changing the average service time, etc.

Poisson Arrivals

For the most part, arrivals do not occur at regular time intervals but tend to be grouped or scattered in an uneven manner. Individual Poisson arrivals are completely independent one another. The assumption of Poisson behavior of arrivals requires the presence of a constant λ which is independent of time, length of queue, or other random characteristics of the queue. The constant λ represents the rate at which units arrive for service.

Monte Carlo Methods in Queueing

Monte Carlo methods are very useful in queueing problems which are difficult to analyze mathematically. The Monte Carlo method uses random sampling to play a game with a system in which an experiment is simulated. In the problem at hand, the technique consists of simulating the arrival of tankers by the use of random sampling. This method of simulated sampling has some advantages over actual sampling of queue characteristics, in that when done on a digital computer, years of data can be developed in a few minutes and those factors which are subject to control can be manipulated. The effects of adding new piers or new tanks to a terminal system can thus be assessed on paper without actual installation or disruption of services. Saaty[66] gives an example of a Monte Carlo solution in a queueing problem.

Limitations of Queueing Theory

Queueing models should be regarded as approximations which serve primarily as guide lines. Arrivals in a system may vary from Poisson loading, which is frequently assumed, but a Poisson input can be regarded as the limit of disintegration for a scheduled process and thus provides a backstop to keep the solution within bounds. An attempt should always be made to solve the queueing problem at its source, the random input. Any action which can be taken to schedule or control the input will be helpful.

With this brief background in queueing theory and the Monte Carlo method, we can proceed to examine the problems of tanker-terminal systems.

Tanker-Terminal Systems

The similarity between military and industrial tanker-terminal systems becomes apparent when it is noted that each must maintain sufficient working or shipping tankage and pier facilities to handle the loading and discharge of tankers. The refiner maintains such facilities for the input of crude oil as well as additional tankage and pier facilities for the output of refined products. Moreover, the refiner must maintain additional tankage for seasonal storage of products such as heating oil. Tankage, comparable to the seasonal tankage operated by the refiner, is operated by the military in that extensive tankage is maintained at military terminals as reserve storage for emergency use.

The similarity of tanker and barge lifting of products from military terminals and refineries is less notable but still generally comparable for purposes of this application. In the servicing (loading and discharge) of tankers, where pier and pumping facilities are limited, delays may occur which are characteristic of queueing situations. The arrival and servicing rates of tankers becomes a significant part of the problem. In most cases, the queueing situation is not believed acute at military terminals, but the military system is affected in that tankers arriving at refineries to lift military products become a part of the queueing problem at the refinery thus having some effect on arrivals at military terminals. In times of emergency this problem could become significant.

Study on Shipping Tankage

Although some attention has been given to the scheduling of tanker movements as transportation and linear programming problems, the published articles on tanker-terminal systems are extremely limited. An article by J. C. Dickson[67] describes an operations research effort aimed at determining the optimal volume of shipping or working tankage at a refinery to handle the loading of tankers. The system described includes the operation of additional seasonal tankage which is normally used for holding heating oil in storage to meet seasonal demands. The system treated by Dickson is considered sufficiently similar to the military system to illustrate the

approach to the problem and the basic factors involved. The model developed by Dickson has been altered somewhat in order to fit the military system more closely and is discussed here.

Description of the Problem

Since tankers do not arrive at fixed intervals and do not load or discharge a fixed quantity of petroleum products, terminals and refineries must maintain shipping or working tanks to absorb the fluctuations experienced in input and output. The consequences of insufficient tankage may be delay of tankers, inability to load or accept a tanker, etc. In the case of the refiner, it may also mean the necessity for altering the entire production schedule of the refinery.

The problem as seen by Dickson was one of minimizing costs. The costs considered to be pertinent to the problem were:

1. Cost of delaying tankers
2. Operating cost of working tanks plus carrying costs on these tanks
3. Cost of reducing input
4. Cost of pumping product to or from reserve storage
5. Carrying costs on product held in inventory

A schematic representation of the system altered to correspond to the military petroleum terminal operation is shown in Figure 12. The symbols used are listed in Table 3.

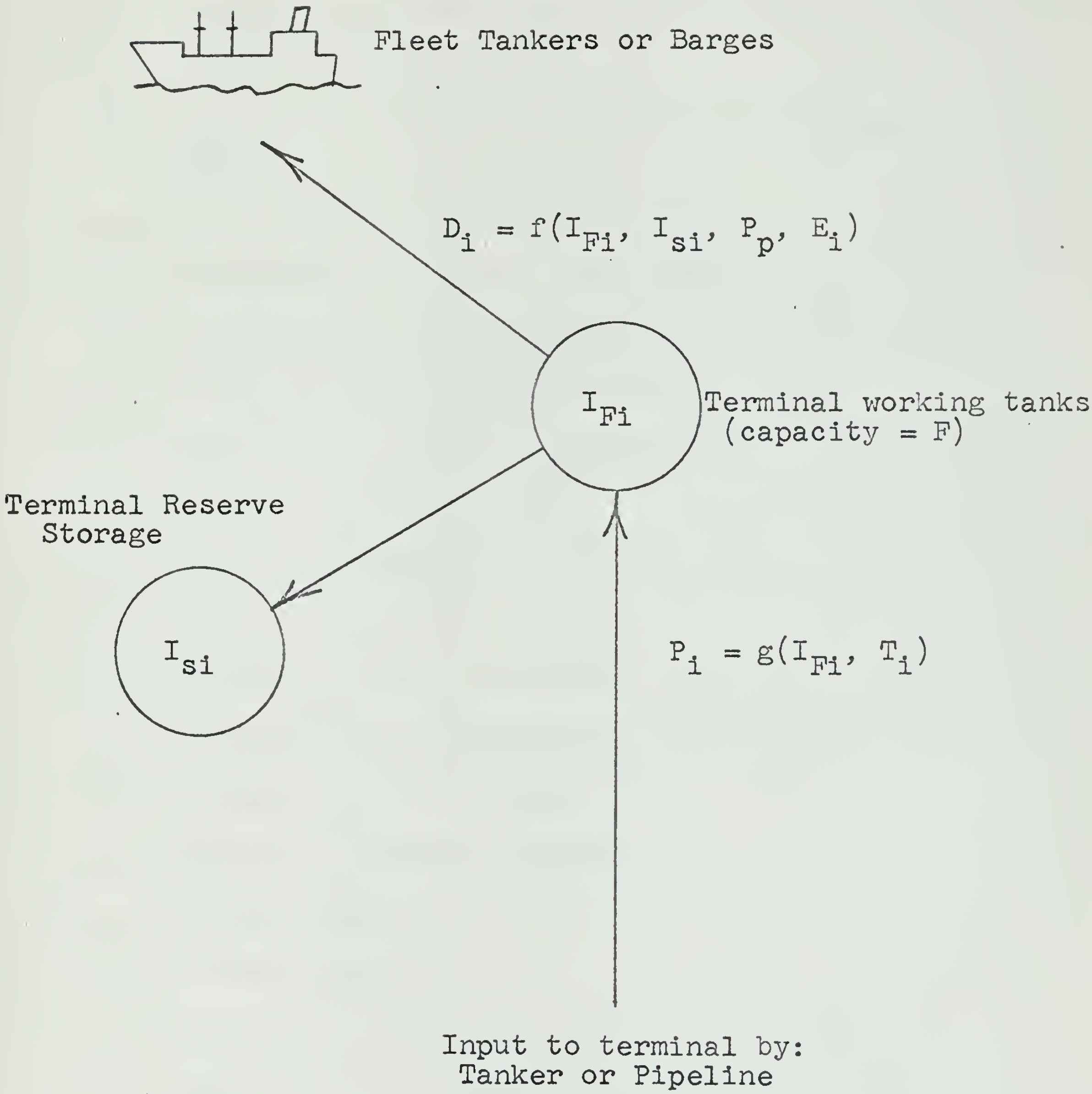

Figure 12

Tanker - Terminal System

Table 3

Explanation of Symbols

i = Day of year being considered

D_i = Demand for product, day i

P_i = Input of product, day i

T_i = Transfer of product from working to storage tanks, day i

I_{Fi} = Inventory in working tanks, day i

I_{Si} = Inventory in reserve tanks, day i

S_i' = Maximum allowable inventory in reserve tanks, day i

C_{Tij} = Cargo size of j^{th} tanker, day i

C_{Bij} = Cargo size of the j^{th} barge, day i

N_{Ti} = Number of tankers, day i

N_{Bi} = Number of barges, day i

M_L = Expected number of tankers

M_B = Expected number of barges

μ_T = Average cargo of tankers

μ_B = Average cargo of barges

T_{max} = Maximum allowable transfer rate

P_{max} = Maximum input

P_p = Planned input

C_c = Upper decision limit for reducing input

$C\mu$ = Upper decision limit for transfer to reserve storage

C_L = Lower decision limit for transfer from reserve storage

C_R = Lower decision limit for input increase

F = Capacity of working tanks

S = Capacity of reserve storage tanks

E_i = Random variable

Analysis of the Problem

The requirement or demand for product on day, D_i, was considered a function of the inventory in the tanks, planned input, and a random variable representing the variations in the arrival of tankers.

The input to the system on day i is designated as P_i and is considered a function of the inventory in the working tanks and the product to be transferred to the reserve storage tanks. T_i, the product transferred to reserve storage on day i, is related to the inventory in both the working and reserve storage tanks.

A few of the functions developed by Dickson for P_i, T_i and D_i are listed here simply to indicate the nature of some of the restrictions utilized in the model.

Daily Input, P_i

$P_i = P_p$ Daily input is equal to planned input if the following conditions exist:

(1) $C_R < I_{Ti} < C_c$; i.e. inventory I_{Fi} in the working tanks must be between the limits beyond which it becomes necessary to increase or decrease input.

(2) $I_{Fi} + P_p - T_i \leqq F$; i.e. planned input will not overflow the tanks after transferring product to reserve storage.

Daily Transfer, T_i

$T_i = T_{i,max}$. Transfer to reserve storage at

maximum transfer rate when the following conditions exist:

(1) $I_{Fi} \geqq C_{\mu}$; i.e. when the inventory in the shipping tank exceeds the upper control limit.

(2) $S'_{i} \geqq T_{max} + T_{si}$; i.e. pumping T_{max} barrels into tanks containing T_{si} barrels will not exceed the allowable inventory S'_{i}.

Daily Demand

The functions defining demand involve the arrival rate of tankers and the size of the cargo lifted. In Dickson's model the arrival rate of tankers was assumed to be a Poisson distribution after study of actual arrival records. A provision was included which was designed to consider the large tankers as scheduled within certain limits but which would require the arrivals of small tankers to follow the above mentioned Poisson distribution.

Solution Method

With the foregoing functions defined, Dickson indicates that inventories on each successive day can be obtained from material balances. Having this information and the delay time on tankers, operating costs are calculated.

The values sought from the model of the system were the optimal volume of shipping or working tankage and the control limits: C_c and C_{μ} which are the upper limits on reducing input and upper limits for transfer to storage; and C_L and C_R which are the lower decision limits on transfer to storage and increasing input respectively.

The method employed to obtain a solution was the Monte Carlo technique which was used to simulate the arrival of tankers and permitted the model to be played rapidly on a computer for a period equivalent to about 40 years. The model was thus used to estimate operating costs while various values were tried for some of the variables subject to control. By averaging the operating costs over several years, a comparatively good estimate of operating costs was anticipated.

Results and Limitations

The results obtained by Dickson contained a sizeable error which was attributed primarily to the assumption of Poisson arrival rates for the tankers. The possibility of an extreme number of tankers arriving on a particular day was eliminated by placing limits on the maximum possible value in the Poisson distribution for tanker arrivals. However, there was no limit on the number of consecutive days without an arrival. This error, and probably others, contributed to the sizeable error in operating costs.

In reviewing the study, Dickson found that because tankage is normally constructed in large increments of 50,000 to 100,000 barrels per tank, the exact value of the shipping tankage was not required and the solution was sufficiently accurate to allow selection of the optimum tankage. The values of the decision limits, for transferring product to or from storage and increasing or decreasing input, were found to be within the required range of accuracy.

Dickson indicates that in any subsequent model, limits should be placed on the number of arrivals on any day and the number of days with no arrivals. He also indicates that future studies should include some type of "anticipator mechanism" since tanker arrivals are usually known two or three days in advance in actual cases.

It is considered by the present author that this method is worthy of study and it is hoped that the indicated modifications would prove adequate to the military problem. It is not likely that a general model could be developed to apply to all terminal situations. The problem of incorrectly assumed Poisson distribution of tanker arrivals can surely be corrected. Tanker arrivals are probably best described as intermediate between Poisson and scheduled. There are indications that tanker arrivals follow more closely the distribution pattern of servically correlated arrivals[68].

CHAPTER IV

APPLICATION OF OPTIMIZATION TECHNIQUES TO MILITARY PETROLEUM PURCHASING POLICIES

As the nation's largest single purchaser and consumer of petroleum products, the Defense Department is vitally interested in the problem of procurement at minimum cost. Total petroleum purchases by the Defense Petroleum Supply Agency were slightly in excess of one billion dollars during the fiscal year 1961. Relying principally on competitive bidding to secure low prices, the Defense Petroleum Supply Agency consistently obtains prices well below those in commercial markets.

The discussion and analysis of purchasing policy presented in this chapter is intended to develop an avenue of approach to some of the basic factors involved in competitive bidding in general and petroleum procurement in particular. It is hoped that an analysis of such factors will serve to indicate those areas where some of the optimization techniques presented in this thesis may be utilized.

Competitive Bidding

The objective of competitive bidding is to purchase a specified product at minimum cost. In the process of preparing a bid, a potential contractor must exercise his strategy in competition with that of his rivals. Although the bidder carries a cushion of profit in his bid, the bid price is, nevertheless, tied to his costs.

If because of some technological development all bidders are able to reduce their costs, it is reasonable to assume that bid prices would also be reduced as long as sufficient competition existed. Similarly, if some of the restrictions placed upon the bidder by the purchaser are altered, a corresponding change in bid prices may be expected. Some of the restrictions placed upon the bidder in the conditions set forth in the invitation for bids are: the type of contract, delivery requirements, specifications, quantity, etc. These do not represent all of the constraints which the bidder must consider, but they are among the major restrictions established by the purchaser. These conditions set forth in the invitation for bids, necessarily have a considerable and perhaps measurable effect on the ultimate price paid by the purchaser.

Analysis of Constraints

In seeking to minimize ultimate purchase prices, the objective of the buyer becomes one of choosing the constraints set forth in the invitation for bids in such a manner as to fulfill the necessary requirements and to permit the bidders to minimize their costs.

While there are a number of conditions set forth in the invitation for bids which act as constraints, there are three which appear to be most important and which may be subject to some degree of control by the purchaser.

The three constraints to be discussed are:

1. Material specifications
2. Quantity
3. Timing of Procurement

Material Specifications

It is recognized that specifications for items to be purchased are rarely subject to the control of the purchasing agency. Specifications, however, do represent a major part of the cost of any petroleum item. This becomes evident when the price paid for each additional octane number is noted. The problem of optimizing specifications is not a simple one. Fuel specifications for an individual aircraft may be determined, but selecting an optimum fuel specification for a wide range of aircraft would be an extremely difficult linear programming problem. Any attempt to determine optimum specifications must consider not only the minimum standards for successful operation, but safety factors and the consequences of failure. Fortunately, most ships and aircraft are designed to fit existing fuels so the problem of optimum specifications may not be real. The analysis of optimum specifications is best left to persons qualified in this area.

Quantity

Military petroleum contracts, for the most part, are written as indefinite quantity contracts under which the government contracts for an estimated quantity but is only

obligated to order a token amount. Under such an arrangement, the government protects itself where fluctuations in requirements occur. The price increase, if any, which the government pays for the uncertainty resting with the contractor is small compared to the advantage gained. Usually the quantity ordered under the contract very nearly equals the amount initially estimated and probably no price differential exists.

Since most military petroleum contracts provide for delivery on an as-required basis, and since the requirements of any given terminal or station are reasonably constant over an extended period, the total quantity contracted for is roughly proportional to the length of the contract period. The constraint placed on the bidders in terms of quantity can thus be said to be the length of the contract period.

Optimum Length of Contract

Numerous military petroleum contracts have a contract life of six months. This is believed to be relatively short. Before discussing possible methods of determining the optimum length of a contract period, let us consider the effects of a change in the length of the contract.

If the contract period were lengthened, inventory costs for the government would remain essentially unchanged since we are considering that petroleum products are ordered only as required. Similarly, inventory costs for the refiner would probably not change appreciably since the volume per

unit time required by the government would remain substantially unchanged. However, with longer term contracts, the refiner could conceivably reduce his costs in the areas of equipment utilization, production scheduling, and possibly in the cost of raw materials. If our assumption is correct, reduced costs to the refiner should result in lower prices to the government.

Let us consider the position of a refiner bidding on one million barrels of 115/145 aviation gasoline. To make this product requires use of an alkylation unit which will be ulilized at, perhaps, 80% of capacity if a contract is secured but at only 50% of capacity without a contract. On the basis of a one year contract it may be that the refiner can operate the alkylation unit to advantage, while with only a six month contract his costs must be spread over the shorter period. To fulfill a short term contract, the refiner may even find it cheaper to buy alkylate from another source and transport it to his refinery. In either event, the cost to the refiner appears to increase under the short term contract.

Not all of the factors to be considered indicate longer term contracts are in order. The fluctuations in the market in terms of demand and price may cause commercial prices to drop while the government still holds a contract at a higher price. Such situations cannot always be predicted. It is also true that bidders sometimes shave their prices under that of the successful bidder on the previous contract. Such

practises are not to be expected over an extended period, however, and the refiners cost must ultimately be the basis of his bid.

The solution of such problems must await the development of models, collection of data and application of a great deal of theory. Linear programming models of refinery processes and the refining industry have been developed as has been shown. Such models contain assumptions which make the determination of an optimum length of contract inaccurate. Nonetheless, it is considered that an indication of the direction in which to move could be determined.

Timing of Procurement

Certain fuels such as heating oil are subject to seasonal demands which result in fluctuations in market prices. In such cases, the timing of a solicitation for bids may be important. If purchases can be made at times when demand and price are at their seasonal lows, some saving can normally be expected. No extended discussion of seasonal purchasing will be attempted here, however, it is considered pertinent to note that the optimal time to purchase one product is not likely to be optimal for another product.

Use of Optimization Techniques

All of the foregoing is best reduced to quantitative terms. The solution of the problems presented would involve the use of several of the basic techniques of linear programming, inventory theory, game theory, and others not

discussed in this thesis. Existing linear programming models of refinery systems might be adapted for use in determining solutions. Extensive data would be required and the cost of such an effort would be substantial.

When it is realized that a reduction in the average purchase price of military petroleum products by as little as one per cent would result in an annual cost reduction of about ten million dollars, the potential of such a project becomes apparent.

The thoughts expressed in this chapter are not those of an authority in the field of military petroleum procurement, and there may be ramifications not known to this author. Much additional study is required before any such project could be initiated.

CHAPTER V

SUMMARY AND CONCLUSION

In this thesis an attempt has been made to survey some of the modern analytical and objective methods for optimizing decisions. The optimization techniques presented are applicable to problems encountered in the operation of military petroleum systems.

The specific techniques covered include: linear programming, game theory, and inventory theory. Certain aspects of queueing theory and the Monte Carlo method were described briefly in connection with a problem involving optimization of tanker-terminal systems.

Sample problems in the area of military petroleum management were developed and solved utilizing some of the methods presented. Problems known to be of current concern to military petroleum management, were discussed with the intention of developing approaches to possible solutions.

Time was not available to develop computer programs for the methods presented. While some problems can be solved by simple arithmetic, it is considered that use of a digital computer is necessary to make significant use of the more sophisticated techniques such as linear programming.

Some of the problems considered represent extensions of existing methods which could be applied directly or which may be found in current use by military supply activities. Others will require additional investigation before the problem can be precisely formulated.

The necessity for precise problem formulation should be emphasized for it is central to the philosophy of modern methods for the optimization of operational systems. One is dealing with problems which must be reduced to precise mathematical form, and a great amount of ingenuity and creativeness is necessary to translate a problem into the terms of a meaningful mathematical model.

The task is difficult on two accounts. First, one must express ideas in mathematical terms which heretofore have not been regarded as quantitative. Second, the formulation must be in a form sufficiently tractable that a numerical solution is feasible. It is for this reason that a large part of the effort of this thesis was devoted to problem formulation and the development of analogies between systems which have been analyzed successfully and systems as yet unanalyzed within the military petroleum supply system.

In conclusion, it is hoped that the material contained herein will be of value to those concerned with the management and analysis of the operation of the military petroleum supply system, and will serve as a basis for further study.

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